Diamond trees, forests, and the exponentiation theorem

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Outline of this talk

- The Itô decomposition formula of Alòs
- Diamond and dot notation
- Stochasticity
- Trees and forests
- The Exponentiation Theorem
- Explicit computations in the rough Heston model

The Itô decomposition formula of Alòs

Following Elisa Alòs in [Alò12], let $X_t = \log S_t/K$ and consider the price process

$$dX_t = \sigma_t\, dZ_t - \frac{1}{2}\,\sigma_t^2\, dt.$$

Now let $H(x,w)$ be some function that solves the Black-Scholes equation.

- Specifically,

$$-\partial_w H(x,w) + \frac{1}{2}\left(\partial_{xx} - \partial_x\right) H(x,w) = 0$$

  which is of course the gamma-vega relationship.
  - Note in particular that ∂_x and ∂_w commute when applied to a solution of the Black-Scholes equation.

Now, define $w_t(T)$ as the integral of the expected future variance:

$$w_t(T) := \mathbb{E}\left[\int_t^T \sigma_s^2 ds \middle| \mathcal{F}_t\right].$$

Notice that

$$w_t(T) = M_t - \int_0^t \sigma_s^2 ds,$$

where the martingale $M_t := \mathbb{E}\left[\int_0^T \sigma_s^2 ds \middle| \mathcal{F}_t\right]$. Then it follows that

$$dw_t(T) = -\sigma_t^2\, dt + dM_t.$$

Applying Itô's Lemma to $H_t := H(X_t, w_t(T))$, taking conditional expectations, simplifying using the Black-Scholes equation and integrating, we obtain

Theorem (The Itô Decomposition Formula of Alòs)

$$\begin{aligned}\mathbb{E}\left[H_T \middle| \mathcal{F}_t\right] &= H_t + \mathbb{E}\left[\int_t^T \partial_{xw} H_s \, d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_t^T \partial_{ww} H_s \, d\langle M, M\rangle_s \middle| \mathcal{F}_t\right]. \qquad (1)\end{aligned}$$

- Note in particular that (1) is an *exact* decomposition.

Freezing derivatives

Freezing the derivatives in the Itô decomposition formula (1) of Alòs gives us the approximation

$$\begin{aligned}\mathbb{E}\left[H_T\middle|\mathcal{F}_t\right] &\approx H_t+\mathbb{E}\left[\int_t^T d\langle X,M\rangle_s\middle|\mathcal{F}_t\right]\partial_{xw}H_t\\&\quad+\frac{1}{2}\mathbb{E}\left[\int_t^T d\langle M,M\rangle_s\middle|\mathcal{F}_t\right]\partial_{ww}H_t\\&= H_t+(X\diamond M)_t(T)\cdot H_t+\frac{1}{2}(M\diamond M)_t(T)\cdot H_t.\end{aligned}$$

Remark

The essence of the Exponentiation Theorem is that we may express $\mathbb{E}\left[H_T\middle|\mathcal{F}_t\right]$ as an exact expansion consisting of infinitely many terms, with derivatives in each such term frozen.

Diamond and dot notation

Let A_t and B_t be continuous semimartingales (here some combinations of X and M). Then

$$(A \diamond B)_t(T) = \mathbb{E}\left[\int_t^T d\langle A, B\rangle_s \middle| \mathcal{F}_t\right].$$

When $(A \diamond B)_t(T)$ appears before some solution H_t of the Black-Scholes equation, the dot $\cdot$ is to be understood as representing the action of ∂_x and ∂_w applied to H_t.

So for example

$$(X \diamond M)_t(T) \cdot H_t \quad = \quad \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{xw} H_t$$

and so on.

Diamond functionals as covariances

- Diamond (or autocovariance) functionals are intimately related to conventional covariances.

Lemma

Let A and B be continuous martingales in the same filtered probability space. Then

$$(A \diamond B)_t(T) = \mathbb{E}\left[A_T B_T \middle| \mathcal{F}_t\right] - A_t\, B_t = \mathrm{cov}\left[A_T, B_T \middle| \mathcal{F}_t\right].$$

- By finding the appropriate martingales, it is thus always possible to re-express autocovariance functionals in terms of covariances of terminal quantities. For example, it is easy to show that $(M \diamond M)_t(T) = \mathrm{var}\left[\langle X \rangle_T \middle| \mathcal{F}_t\right]$.

Autocovariance functionals vs covariances

- Covariances are typically easy to compute using simulation.
- Diamond functionals are expressible directly in terms of the formulation of a model in forward variance form.

Conditional variance of X_T

Consider

$$F_t = X_t^2 + w_t(T)\,(1 - X_t) + \frac{1}{4}\,w_t(T)^2.$$

- $F(x, w)$ satisfies the Black-Scholes equation and $F_T = X_T^2$.
  - $\partial_{x,w} F = -1$ and $\partial_{w,w} F = \frac{1}{2}$.
- Plugging into the Decomposition Formula (1) gives

$$\begin{aligned}
\mathbb{E}\left[X_T^2 \middle| \mathcal{F}_t\right] &= w_t(T) + \frac{1}{4}\,w_t(T)^2 - \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \\
&\quad + \frac{1}{4}\,\mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \\
&= w_t(T) + \frac{1}{4}\,w_t(T)^2 \\
&\quad - (X \diamond M)_t(T) + \frac{1}{4}\,(M \diamond M)_t(T).
\end{aligned}$$

Volatility stochasticity

We can rewrite this as

Lemma

$$\zeta_t(T) := \mathrm{var}[X_T|\mathcal{F}_t] - w_t(T) = -(X \diamond M)_t(T) + \frac{1}{4}(M \diamond M)_t(T).$$

- Recall that in a stochastic volatility model, the variance of the terminal distribution of the log-underlying is not in general equal to the expected quadratic variation.
  - In the Black-Scholes model of course $\zeta_t(T) = 0$.
- We call the difference $\zeta_t(T)$ *volatility stochasticity* or just *stochasticity*.

Model calibration

Once again, stochasticity is given by

$$\zeta_t(T) = -(X \diamond M)_t(T) + \frac{1}{4}(M \diamond M)_t(T).$$

- The LHS may be estimated from the volatility surface using the spanning formula.
    - $\zeta_t(T)$ is a tradable asset for each T.
    - We get a matching condition for each expiry T_i, $i \in \{1, ..n\}$.
- The RHS may typically be computed in a given model as a function of model parameters.
    - If so, we would be able to calibrate such a model directly to tradable assets with no need for any expansion.

$\zeta_t(T)$ directly from the smile

Let

$$d_\pm(k) = \frac{-k}{\sigma_{\mathrm{BS}}(k,T)\sqrt{T}} \pm \frac{\sigma_{\mathrm{BS}}(k,T)\sqrt{T}}{2}$$

and following Fukasawa, denote the inverse functions by $g_\pm(z) = d_\pm^{-1}(z)$. Further define

$$\sigma_-(z) = \sigma_{\mathrm{BS}}(g_-(z),T)\,\sqrt{T}.$$

In terms of the implied volatility smile, it is a well-known corollary of Matytsin's characteristic function representation in [Mat00], that

$$w_t(T) = \int dz\, N'(z)\, \sigma_-^2(z) =: \bar{\sigma}^2.$$

Similarly, we can show that

$$\zeta_t(T) = \frac{1}{4} \int N'(z)\, \left[\sigma_-^2(z) - \bar{\sigma}^2\right]^2 dz + \frac{2}{3} \int N'(z)\, z\, \sigma_-^3(z)\, dz.$$

- We may thus in principle use stochasticity to calibrate any given model.
  - In practice, we need a good parameterization of the implied volatility surface (see VolaDynamics later).
  - Whether or not market implied stochasticity is robust to the interpolation and extrapolation method is still to be explored.

Forward variance models

Following [BG12], consider the model

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t}\,\left\{\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}\right\} \\ d\xi_t(u) &= \lambda(t,u,\xi_t)\, dW_t. \end{aligned} \tag{2}$$

where $v_t = \sigma_t^2$ denotes instantaneous variance and the $\xi_t(u) = \mathbb{E}\left[\, v_u \middle| \mathcal{F}_t\right], u \in [t, T]$ are forward variances.

- To expand such a model, we scale the volatility of volatility function $\lambda(\cdot)$ so that $\lambda \mapsto \epsilon\,\lambda$. Setting $\epsilon = 1$ at the end then gives the required expansion.

The Bergomi-Guyon expansion

- According to equation (13) of [BG12], in diamond notation, the conditional expectation of a solution of the Black-Scholes equation satisfies

$$
\begin{aligned}
& \mathbb{E}\left[H_T \middle| \mathcal{F}_t \right] \\
= \; & \Big\{ 1 + \epsilon \, (X \diamond M)_t + \frac{\epsilon^2}{2} \, (M \diamond M)_t \\
& \quad + \frac{\epsilon^2}{2} \, \left[(X \diamond M)_t \right]^2 + \epsilon^2 \, (X \diamond (X \diamond M))_t + \mathcal{O}(\epsilon^3) \Big\} \cdot H_t
\end{aligned}
$$

- We notice that

$$\mathbb{E}\left[H_T \middle| \mathcal{F}_t\right] = \exp\left\{\epsilon\,(X \diamond M)_t + \frac{\epsilon^2}{2}\,(M \diamond M)_t + \epsilon^2\,(X \diamond (X \diamond M))_t + \mathcal{O}(\epsilon^3)\right\} \cdot H_t,$$

  the exponential of a sum of "connected diagrams".
- Motivated by exponentiation results in physics, we are tempted to see if something like this holds to all orders.

Trees

- Terms such as $(X \diamond M)$, $(M \diamond M)$ and $X \diamond (X \diamond M)$ are naturally indexed by trees, each of whose leaves corresponds to either X or M.
- We end up with diamond trees reminiscent of Feynman diagrams, with analogous rules.

Forests

Definition

Let $\mathbb{F}_0 = M$. Then the higher order forests $\mathbb{F}_k$ are defined recursively as follows:

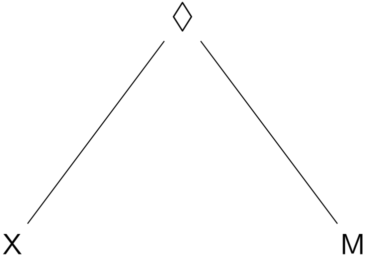
$$\mathbb{F}_k = \frac{1}{2} \sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2} \, \mathbb{F}_i \diamond \mathbb{F}_j + X \diamond \mathbb{F}_{k-1}.$$

The first few forests

Applying this definition to compute the first few terms, we obtain

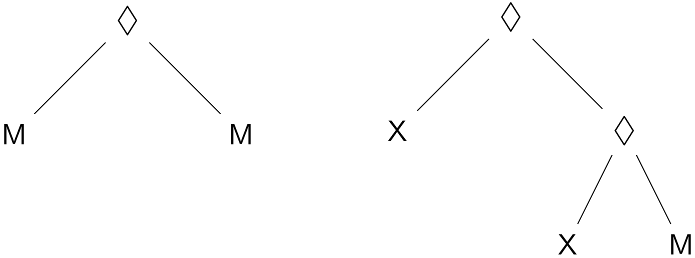
$$\begin{aligned}
\mathbb{F}_0 &= M \\
\mathbb{F}_1 &= X \diamond \mathbb{F}_0 = (X \diamond M) \\
\mathbb{F}_2 &= \frac{1}{2}(\mathbb{F}_0 \diamond \mathbb{F}_0) + X \diamond \mathbb{F}_1 = \frac{1}{2}(M \diamond M) + X \diamond (X \diamond M) \\
\mathbb{F}_3 &= (\mathbb{F}_0 \diamond \mathbb{F}_1) + X \diamond \mathbb{F}_2 \\
&= M \diamond (X \diamond M) + \frac{1}{2} X \diamond (M \diamond M) + X \diamond (X \diamond (X \diamond M))
\end{aligned}$$

The first forest $\mathbb{F}_1 = X \diamond M$



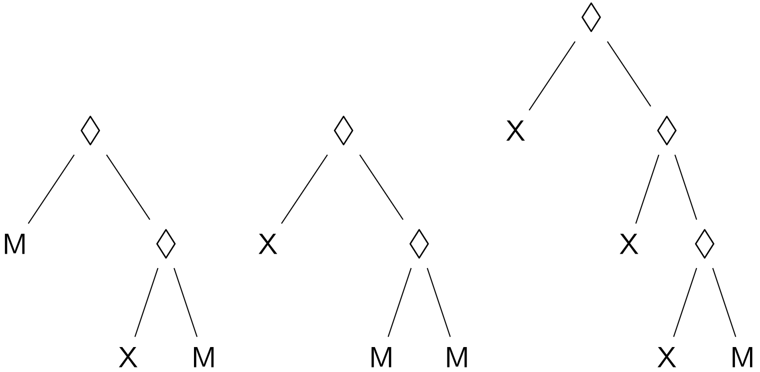
The second forest $\mathbb{F}_2$

$$\mathbb{F}_2 = \frac{1}{2}(M \diamond M) + X \diamond (X \diamond M)$$



The third forest $\mathbb{F}_3$

$$\mathbb{F}_3 = M \diamond (X \diamond M) + \frac{1}{2} X \diamond (M \diamond M) + X \diamond (X \diamond (X \diamond M))$$

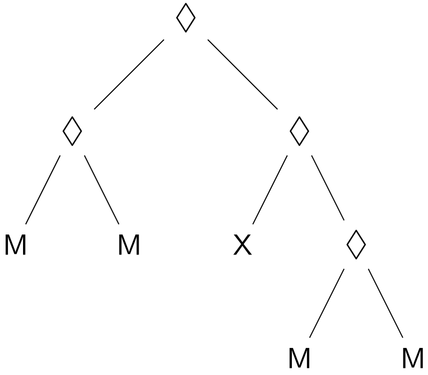


Simple diamond rules

- For $k > 0$, the kth forest $\mathbb{F}_k$ contains all trees with $k+2$ leaves where X is counted as a single leaf, and M as a double leaf.
- Prefactor computation:
  - Work from the bottom up.
  - If child subtrees immediately below a diamond node are identical, carry a multiplicative factor of $\frac{1}{2}$.

Example: One tree in $\mathbb{F}_7$

$$\frac{1}{4}(M \diamond M) \diamond (X \diamond (M \diamond M))$$



The Exponentiation Theorem

The following theorem proved in [AGR2017] follows from (more or less) a simple application of Itô's Lemma and the Itô decomposition formula of Alòs.

Theorem

Let H_t be any solution of the Black-Scholes equation such that $\mathbb{E}\left[H_T \middle| \mathcal{F}_t\right]$ is finite and the integrals contributing to each forest $\mathbb{F}_k, k \geq 0$ exist. Then

$$\mathbb{E}\left[H_T \middle| \mathcal{F}_t\right] = e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H_t.$$

If H_t is a characteristic function

Consider the Black-Scholes characteristic function

$$\Phi_t^T(a) = e^{\mathrm{i}\, a\, X_t - \frac{1}{2} a\,(a+\mathrm{i})\, w_t(T)}$$

which satisfies the Black-Scholes equation.

- Applying $\mathbb{F}_k$ to Φ just multiplies Φ by some deterministic factor.
- Then
$$e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot \Phi_t^T(a) = e^{\sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a)}\, \Phi_t^T(a)$$
  where $\tilde{\mathbb{F}}_k(a)$ is $\mathbb{F}_k$ with each occurrence of ∂_x replaced with $\mathrm{i}\, a$ and each occurrence of ∂_w replaced with $-\frac{1}{2}\, a\,(a+\mathrm{i})$.

Then from the Exponentiation Theorem, we have the following lemma.

Lemma

Let

$$\varphi_t^T(a) = \mathbb{E}\left[e^{i\,a\,X_T} \middle| \mathcal{F}_t \right]$$

be the characteristic function of the log stock price. Then

$$\varphi_t^T(a) = e^{\sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a)}\, \Phi_t^T(a).$$

Corollary

The cumulant generating function (CGF) is given by

$$\psi_t^T(a) = \log \varphi_t^T(a) = \mathrm{i}\,a\,X_t - \frac{1}{2} a\,(a + \mathrm{i})\,w_t(T) + \sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a).$$

Variance and gamma swaps

The variance swap is given by the fair value of the log-strip:

$$\mathbb{E}\left[X_T \middle| \mathcal{F}_t\right] = (-\mathrm{i})\, \psi_t^{T\,\prime}(0) = X_t - \frac{1}{2}\, w_t(T)$$

and the gamma swap (wlog set $X_t = 0$) by

$$\mathbb{E}\left[X_T\, e^{X_T} \middle| \mathcal{F}_t\right] = -\mathrm{i}\, \psi_t^{T\,\prime}(-\mathrm{i}).$$

Remark

The point is that we can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.

The gamma swap

It is easy to see that only trees containing a single M leaf will survive in the sum after differentiation when $a = -\mathrm{i}$ so that

$$\sum_{k=1}^{\infty} \tilde{\mathbb{F}}'_k(-\mathrm{i}) = \frac{\mathrm{i}}{2} \sum_{k=1}^{\infty} (X\diamond)^k M$$

where $(X\diamond)^k M$ is defined recursively for $k > 0$ as $(X\diamond)^k M = X \diamond (X\diamond)^{k-1} M$. Then the fair value of a gamma swap is given by

$$\mathcal{G}_t(T) = 2\,\mathbb{E}\left[X_T\, e^{X_T} \middle| \mathcal{F}_t \right] = w_t(T) + \sum_{k=1}^{\infty} (X\diamond)^k M. \tag{3}$$

Remark

Equation (3) allows for explicit computation of the gamma swap for any model written in forward variance form.

The leverage swap

We deduce that the fair value of a leverage swap is given by

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - w_t(T) = \sum_{k=1}^{\infty} (X\diamond)^k M. \tag{4}$$

- The leverage swap is expressed explicitly in terms of covariance functionals of the spot and vol. processes.
  - If spot and vol. processes are uncorrelated, the fair value of the leverage swap is zero.

An explicit expression for the leverage swap!

Leverage swap from the smile

- Define
$$\sigma_\pm(z) = \sigma_{\mathrm{BS}}(g_\pm(z),T)\,\sqrt{T}.$$
where $g_\pm$ are the Fukasawa inverse functions introduced earlier.
- Then the gamma swap may be estimated from the smile using
$$\mathcal{G}_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_+^2(z).$$
- And as before, the variance swap is given by
$$w_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_-^2(z).$$
- Recall that $\mathcal{L}_t(T) = \mathcal{G}_t(T) - w_t(T)$.

Skewness

As is well-known, the first three central moments are easily computed from cumulants by differentiation. For example, skewness is given by

$$\begin{aligned}
\mathcal{S}_t(T) &:= \mathbb{E}\left[(X_T - \bar{X}_T)^3 \middle| \mathcal{F}_t \right] \\
&= (-\mathrm{i})^3 \, \psi_t^{T\,'''}(0) \\
&= -\frac{3}{2}\,(M \diamond M)_t(T) - \frac{3}{8}\,(M \diamond (M \diamond M))_t(T) \\
&\qquad + \frac{3}{2}\,(M \diamond (X \diamond M))_t(T) + 3\,(X \diamond M)_t(T) \\
&\qquad + \frac{3}{4}\,(X \diamond (M \diamond M))_t(T) - 3\,(X \diamond (X \diamond M))_t(T).
\end{aligned} \tag{5}$$

An explicit expression for skewness!

The Bergomi-Guyon smile expansion

- The Bergomi-Guyon (BG) smile expansion (Equation (14) of [BG12]) reads

$$\sigma_{\mathrm{BS}}(k,T) = \hat{\sigma}_T + \mathcal{S}_T\, k + \mathcal{C}_T\, k^2 + \mathcal{O}(\epsilon^3)$$

  where the coefficients $\hat{\sigma}_T$, $\mathcal{S}_T$ and $\mathcal{C}_T$ are complicated combinations of trees such as $X \diamond M$.
- As we have seen, such trees are formally easily computable in any stochastic volatility model written in forward variance form.
- The beauty of the BG expansion is that in some sense, it yields direct relationships between the smile and autocovariance functionals.

Bergomi-Guyon to higher order

- We can extend the Bergomi-Guyon expansion to any desired order using our formal expression for the CGF in terms of forests.
- To second order, ATM total variance is given by

$$\sigma^2_{\mathrm{BS}}(k,T)\,T = w_t(T) + \epsilon\, a_1(k) + \epsilon^2\, a_2(k) + \mathcal{O}(\epsilon^3) \tag{6}$$

where, with $w_t(T) = w$ for ease of notation,

$$\begin{aligned} a_1(k) &= \left(\frac{k}{w} + \frac{1}{2}\right)(X \diamond M) \\ a_2(k) &= \frac{1}{4}(X \diamond M)^2 \left\{-\frac{5k^2}{w^3} - \frac{2k}{w^2} + \frac{3}{w^2} + \frac{1}{4w}\right\} \\ &\quad + \frac{1}{4}(M \diamond M)\left\{\frac{k^2}{w^2} - \frac{1}{w} - \frac{1}{4}\right\} \\ &\quad + (X \diamond (X \diamond M))\left\{\frac{k^2}{w^2} + \frac{k}{w} - \frac{1}{w} + \frac{1}{4}\right\}. \end{aligned}$$

- As an example of a higher order term, the ATM total variance skew is given to third order by:

$$
\begin{aligned}
&\partial_k \sigma_{\mathrm{BS}}(k,T)^2\, T\Big|_{k=0} \\
= \quad & \frac{X \diamond M}{w} + \frac{X \diamond (X \diamond M)}{w} - \frac{1}{2} \left(\frac{X \diamond M}{w}\right)^2 \\
&+ \frac{3}{4} (X \diamond (X \diamond (X \diamond M)))\, \frac{w-4}{w^2} - (M \diamond (X \diamond M))\, \frac{w+12}{8w^2} \\
&- (X \diamond (M \diamond M))\, \frac{w+12}{16w^2} + (M \diamond M)\,(X \diamond M)\, \frac{w+14}{8w^3} \\
&+ (X \diamond M)^3\, \frac{w-64}{16w^4} - \frac{1}{2} (X \diamond M)\,(X \diamond (X \diamond M))\, \frac{w-14}{w^3} + \ldots
\end{aligned}
$$

Skewness, leverage, stochasticity and the volatility skew

- The explicit expression (5) for skewness applies to any stochastic volatility model written in forward variance form.
- There are numerous references in the literature to the connection between the implied volatility skew and both the skewness and the leverage swap.
- Our explicit expression shows how these three quantities are related.
  - Denoting the ATM implied volatility skew by $\psi_t(T)$, we have from the BG expansion that to lowest order,

$$\psi_t(T) = \sqrt{\frac{w}{T}}\,\frac{1}{2\,w^2}\,(X \diamond M)_t(T)$$

and to lowest order in the forest expansion,

$$\frac{1}{3}\,\mathcal{S}_t(T) = (X \diamond M)_t(T) = \mathcal{L}_t(T) = -\zeta_t(T).$$

The rough Heston model

In the zero mean reversion limit, the rough Heston model of [ER19] may be written as

$$\frac{dS_t}{S_t} = \sqrt{v_t}\,\left\{\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}\right\} = \sqrt{v_t}\, dZ_t$$

with

$$v_u = \xi_t(u) + \frac{\nu}{\Gamma(\alpha)} \int_t^u \frac{\sqrt{v_s}}{(u-s)^{\gamma}}\, dW_s, \quad u \geq t$$

where $\xi_t(u) = \mathbb{E}\left[\, v_u \middle| \mathcal{F}_t \right]$ is the forward variance curve, $\gamma = \frac{1}{2} - H$ and $\alpha = 1 - \gamma = H + \frac{1}{2}$.

The rough Heston model in forward variance form

In forward variance form,

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)} \frac{\sqrt{v_t}}{(u-t)^\gamma} \, dW_t. \tag{7}$$

Remark

(7) is a natural fractional generalization of the classical Heston model which reads, in forward variance form [BG12],

$$d\xi_t(u) = \nu \sqrt{v_t} \, e^{-\kappa\,(u-t)} \, dW_t.$$

Computation of autocovariance functionals

Apart from $\mathcal{F}_t$ measurable terms (abbreviated as 'drift'), we have

$$
\begin{aligned}
dX_t &= \sqrt{v_t}\, dZ_t + \text{drift} \\
dM_t &= \int_t^T d\xi_t(u)\, du \\
&= \frac{\nu}{\Gamma(\alpha)} \sqrt{v_t} \left(\int_t^T \frac{du}{(u-t)^{\gamma}} \right) dW_t \\
&= \frac{\nu\,(T-t)^{\alpha}}{\Gamma(1+\alpha)} \sqrt{v_t}\, dW_t.
\end{aligned}
$$

The first order forest

There is only one tree in the forest $\mathbb{F}_1$.

$$\begin{aligned}
\mathbb{F}_1 = (X \diamond M)_t(T) &= \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t \right] \\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}\left[\int_t^T v_s\,(T-s)^\alpha\,ds \middle| \mathcal{F}_t \right] \\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)} \int_t^T \xi_t(s)\,(T-s)^\alpha\,ds.
\end{aligned}$$

Higher order forests

Define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds\, \xi_t(s)\, (T-s)^{j\,\alpha}.$$

Then

$$\begin{aligned} dI_s^{(j)}(T) &= \int_s^T du\, d\xi_s(u)\, (T-u)^{j\,\alpha} + \text{ drift terms} \\ &= \frac{\nu\,\sqrt{v_s}}{\Gamma(\alpha)}\, dW_s \int_s^T \frac{(T-u)^{j\,\alpha}}{(u-s)^{\gamma}}\, du + \text{ drift terms} \\ &= \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\, \nu\,\sqrt{v_s}\,(T-s)^{(j+1)\,\alpha}\, dW_s + \text{ drift terms.} \end{aligned}$$

With this notation,

$$(X \diamond M)_t(T) = \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, I_t^{(1)}(T).$$

The second order forest

There are two trees in $\mathbb{F}_2$:

$$\begin{aligned}
(M \diamond M)_t(T) &= \mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t \right] \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(s)\,(T-s)^{2\,\alpha}\,ds \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2}\, I_t^{(2)}(T)
\end{aligned}$$

and

$$\begin{aligned}
(X \diamond (X \diamond M))_t\,(T) &= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, \mathbb{E}\left[\int_t^T d\langle X, I^{(1)}\rangle_s \middle| \mathcal{F}_t \right] \\
&= \frac{\rho^2\,\nu^2}{\Gamma(1+2\,\alpha)}\, I_t^{(2)}(T).
\end{aligned}$$

The third order forest

Continuing to the forest $\mathbb{F}_3$, we have the following.

$$
\begin{aligned}
(M \diamond (X \diamond M))_t(T) &= \frac{\rho\,\nu^3}{\Gamma(1+\alpha)\,\Gamma(1+2\,\alpha)}\, I_t^{(3)}(T)\\
(X \diamond (X \diamond (X \diamond M)))_t(T) &= \frac{\rho^3\,\nu^3}{\Gamma(1+3\,\alpha)}\, I_t^{(3)}(T)\\
(X \diamond (M \diamond M))_t(T) &= \frac{\rho\,\nu^3\,\Gamma(1+2\,\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\,\alpha)}\, I_t^{(3)}(T).
\end{aligned}
$$

In particular, we easily identify the pattern

$$
(X\diamond)^k M_t(T) = \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)}\, I_t^{(k)}(T).
$$

The leverage swap under rough Heston

Using (4), we have

$$
\begin{aligned}
\mathcal{L}_t(T) &= \sum_{k=1}^{\infty} (X\diamond)^k M_t(T) \\
&= \sum_{k=1}^{\infty} \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)} \int_t^T du\, \xi_t(u)\,(T-u)^{k\,\alpha} \\
&= \int_t^T du\, \xi_t(u)\, \left\{E_\alpha(\rho\,\nu\,(T-u)^\alpha) - 1\right\} \qquad (8)
\end{aligned}
$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function.

An explicit expression for the leverage swap!

The normalized leverage swap

Given the form of equation (8), it is natural to normalize the leverage swap by the variance swap. We therefore define

$$L_t(T) = \frac{\mathcal{L}_t(T)}{w_t(T)}. \tag{9}$$

In the special case of the rough Heston model with a flat forward variance curve,

$$L_t(T) = E_{\alpha,2}(\rho\,\nu\,\tau^\alpha) - 1,$$

where $E_{\alpha,2}(\cdot)$ is a generalized Mittag-Leffler function. We further define an nth order approximation to $L_t(T)$ as

$$L_t^{(n)}(T) = \sum_{k=1}^{n} \frac{(\rho\,\nu\,\tau^\alpha)^k}{\Gamma(2+k\,\alpha)}.$$

A numerical example

We now perform a numerical computation of the value of the leverage swap using the forest expansion in the rough Heston model with the following parameters, calibrated to the SPX options market as of April 24, 2017:

$$H = 0.0236; \quad \nu = 0.3266; \quad \rho = -0.6510.$$

The leverage swap under rough Heston

In Figure 1, we plot the normalized leverage swap $L_t(T)$ and successive approximations $L_t^{(n)}(T)$ to it as a function of τ.

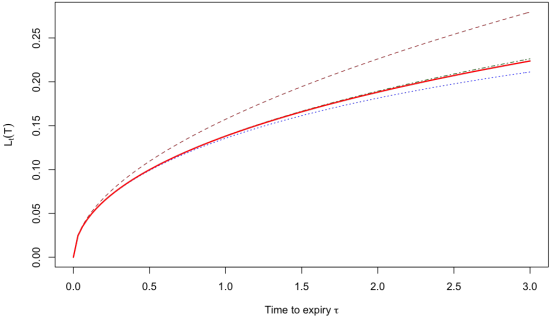


Figure 1: Successive approximations to the (absolute value of) the normalized rough Heston leverage swap. The solid red line is the exact expression $L_t(T)$; $L_t^{(1)}(T)$, $L_t^{(2)}(T)$, and $L_t^{(3)}(T)$ are brown dashed, blue dotted and dark green dash-dotted lines respectively.

The leverage swap under rough Heston

We note that three terms are enough to get a very good approximation to the normalized leverage swap for all expirations traded in the listed market. Moreover, leverage swaps are straightforward to estimate from volatility smiles.

Remark

In practice, (9) can be used for very fast and efficient calibration of the three parameters of the rough Heston model by minimizing the distance between model and empirical normalized leverage swap estimates.

Leverage estimates using VolaDynamics

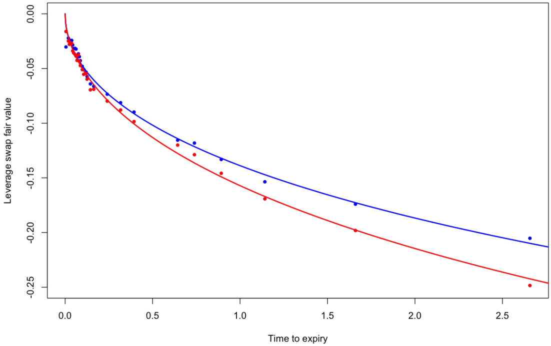


Figure 2: Leverage estimates using the VolaDynamics curves C13PM (blue) and C14PM (red) and their respective rough Heston fits as of 24-Apr-2017. See https://voladynamics.com.

- Whether or not it is possible to robustly estimate model parameters in this way remains an open question.

Summary

- We stated the Itô Decomposition Formula of Alòs.
- We introduced diamond notation.
- We defined trees and forests and showed how to compute all such forests diagrammatically.
- We stated the Exponentiation Theorem.
- We used this theorem to compute various quantities of interest under rough Heston in closed form.

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