Diamond trees, forests, cumulants, and martingales

Jim Gatheral
(joint work with Elisa Alòs, Peter Friz and Radoš Radoičić)

Baruch College
Weissman School of Arts and Sciences
Master of Financial Engineering

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Outline of this talk

- The diamond product
- The $\mathbb{K}$-expansion
  - Trees and forests
  - MGF of the Lévy area
- The $\mathbb{F}$-expansion
- Applications:
  - Leverage swaps
  - The Bergomi-Guyon smile expansion to all orders
  - Computations in the rough Heston model

The diamond product

Definition

Given two continuous semimartingales A, B with integrable covariation process $\langle A, B\rangle$, the diamond product[a] of A and B is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E}_t\left[\langle A, B\rangle_{t,T}\right] = \mathbb{E}_t\left[\langle A, B\rangle_T\right] - \langle A, B\rangle_t,$$

where $\langle A, B\rangle_{t,T} = \langle A, B\rangle_T - \langle A, B\rangle_t$.

[a] Warning. Our diamond product is (very) different from the Wick product.

Diamond products vs covariances

- Diamond products are intimately related to conventional covariances.

Lemma

Let A and B be continuous martingales in the same filtered probability space. Then

$$(A \diamond B)_t(T) = \mathbb{E}_t\left[A_T B_T\right] - A_t\, B_t = \mathrm{cov}\left[A_T, B_T \middle| \mathcal{F}_t\right].$$

- Covariances are typically easy to compute using simulation.
- Diamond products are expressible directly in terms of the dynamics of A and B.

Properties of the diamond product

- Commutative: $A \diamond B = B \diamond A$.
- Non-associative: $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$.
- $A \diamond B$ depends only on the respective martingale parts of A and B.
- $A \diamond B$ is in general not a martingale.

The $\mathbb{K}$-forest expansion

Theorem 1

(i) Let A_T be $\mathcal{F}_T$-measurable with $N \in \mathbb{N}$ finite moments. Then the recursion

$$\mathbb{K}_t^{n+1}(T) = \frac{1}{2}\sum_{k=1}^{n}(\mathbb{K}^k \diamond \mathbb{K}^{n+1-k})_t(T), \quad \forall n > 0 \tag{1}$$

with $\mathbb{K}_t^1(T) := \mathbb{E}_t\left[A_T\right]$ is well-defined up to $\mathbb{K}^N$ and, for $a \in \mathbb{R}$,

$$\log \mathbb{E}_t\left[e^{iaA_T}\right] = \sum_{n=1}^{N}(ia)^n\,\mathbb{K}_t^n(T) + o(|a|^N)$$

which identifies $n! \times \mathbb{K}_t^n(T)$ as the (time t-conditional) n.th cumulant of A_T.

Theorem 1 (cont.)

(ii) If A_T has moments of all orders, we have the asymptotic expansion,

$$\log \mathbb{E}_t \left[e^{iaA_T} \right] \sim \sum_{n=1}^{\infty} (ia)^n \, \mathbb{K}_t^n \quad \text{as } a \to 0 \,. \tag{2}$$

(iii) If A_T has exponential moments, so that its (time t-conditional) mgf $\mathbb{E}_t \left[e^{xA_T} \right]$ is a.s. finite for $x \in \mathbb{R}$ in some neighbourhood of zero, then there exist a maximal convergence radius $\rho = \rho_t(\omega) \in (0, \infty]$ a.s. such that for all $z \in \mathbb{C}$ with $|z| < \rho$,

$$\log \mathbb{E}_t \left[e^{zA_T} \right] = \sum_{n=1}^{\infty} z^n \mathbb{K}_t^n \,. \tag{3}$$

Convergence

- Since the $\mathbb{K}$-expansion is just the cumulant expansion, it inherits the following convergence properties.

Lemma 2

(i) Let A be a real-valued random variable with n moments, $n \in \mathbb{N}$. Then the characteristic function $\xi \mapsto \mathbb{E}\left[e^{i\xi A}\right]$ is n times differentiable at zero and, as $\xi \to 0$,

$$\phi_A(\xi) = \mathbb{E}\left[e^{i\xi A}\right] = \sum_{j=1}^{n} \frac{\kappa_j}{j!}(i\xi)^j + o(|\xi|^n)$$

where $\kappa_j := i^{-j}\phi_A^{(j)}(0)$ is called j.th cumulant of A.

Lemma 2 (cont.)

(ii) Let A be a real-valued random variable with exponential moments by which we mean that the mgf $\mathrm{M}(x) = \mathbb{E}\left[e^{xA}\right]$ is finite in neighbourhood of 0. Then, for x in a (possibly smaller) neighbourhood of 0, in terms of cumulants $\kappa_n := \Lambda^{(n)}(0)$,

$$\log \mathbb{E}\left[e^{xA}\right] = \Lambda(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}\kappa_n \, .$$

This expansion is also valid for $x \in \mathbb{R}$ replaced by (small) enough $z \in \mathbb{C}$.

Multivariate cumulants and martingales

Though the statement and proof of Theorem 1 assume a random variable in $\mathbb{R}$, the extension to $\mathbb{R}^d$ is straightforward.

Theorem 3

Let A_T is be a d-dimensional random variable s.t. $\mathbb{E}_t[e^{x \cdot A_T}]$ is a.s. finite for $x \in \mathbb{R}^d$ in some neighbourhood of zero. Then, for all $z \in \mathbb{C}$ small enough,

$$\log \mathbb{E}_t[e^{z \cdot A_T}] = \sum_{n=1}^{\infty} z^{\otimes n} \cdot \mathbb{K}_t^{(n)} = z_i \mathbb{K}_t^i + z_i z_j \mathbb{K}_t^{i,j} + z_i z_j z_k \mathbb{K}_t^{i,j,k} + \ldots .$$

- Each index $i, j, k, \ldots$ can be associated with a different leaf color in a tree (explanation to follow).

Theorem 3 (cont.)

The $\{\mathbb{K}_t^{(n)}(T) : n \geqslant 1\}$ *satisfy the recursion* $\mathbb{K}_t^{(1)} = \mathbb{E}_t[A_T] \in \mathbb{R}^d$ *and*

$$
\begin{aligned}
\mathbb{K}_t^{(n+1)} &= \frac{1}{2}\sum_{k=1}^{n} \mathbb{E}_t\left[\langle \mathbb{K}^{(k)} \otimes \mathbb{K}^{(n+1-k)}\rangle_{t,T}\right] \\
&= \frac{1}{2}\sum_{k=1}^{n} (\mathbb{K}^{(k)} \diamond \mathbb{K}^{(n+1-k)})_t(T) \in (\mathbb{R}^d)^{\otimes(n+1)}.
\end{aligned}
$$

Comments on Theorem 3

- Here the diamond product for a vector-valued semimartingale, with values in finite-dimensional spaces V, W say, is understood component-wise, giving a $V \otimes W$-valued semimartingale.
    - For instance if $V = (\mathbb{R}^d)^{\otimes k}$, $W = (\mathbb{R}^d)^{\otimes((n+1-k))}$, then $V \otimes W \cong (\mathbb{R}^d)^{\otimes(n+1)}$.
- The first instance of such an expansion, with $d = 2$ and $A_T = (X_T, \langle X \rangle_T)$ and $z = (a, -\frac{1}{2} a)$

$$z \cdot A_T = aX_T - \tfrac{1}{2}a\langle X \rangle_T$$

appeared in [AGR2020], initially posted online in 2017.

Another paper

- Shortly after posting [FGR20] on the arXiv, we were informed by Vincent Vargas that Part (iii) of Theorem 1 was independently proved in [LRV19] in the context of renormalization of the sine-Gordon model in quantum physics.
  - Unaware of [AGR2020], the proof of [LRV19] is in $d = 1$. In turn, unaware of [LRV19], we gave a (different) proof, which applied to the case $d = 2$, and after some reordering, explains the link to [AGR2020].
- We find it remarkable how problems in quantitative finance and quantum physics lead to the same nice mathematics.

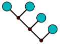
Trees and forests

- The general term $\mathbb{K}_t^n(T)$ in Theorem 1 is naturally written as a linear combination of binary diamond trees[1].
- Hence the terminology $\mathbb{K}$-*forest expansion* for (2) and (3) .
- Specifically, writing Y as a short-hand for $\mathbb{K}_t^1(T)$ we have

$$\mathbb{K}^1 = Y \equiv$$


$$\mathbb{K}^2 = \tfrac{1}{2} Y \diamond Y \equiv \tfrac{1}{2}$$


$$\mathbb{K}^3 = \tfrac{1}{2} (Y \diamond Y) \diamond Y \equiv \tfrac{1}{2}$$


$$\mathbb{K}^4 = \tfrac{1}{2} ((Y \diamond Y) \diamond Y) \diamond Y + \tfrac{1}{8} (Y \diamond Y)^{\diamond 2} \equiv \tfrac{1}{2}$$

$$+ \tfrac{1}{8}$$


$$\cdots$$

[1] Trees stolen from [Hai13]!

Simple diamond tree rules

- For $n \geq 1$, the nth forest $\mathbb{K}^n$ contains all trees with n leaves.
- Prefactor computation:
    - Work from the bottom up.
    - If child subtrees immediately below a diamond node are identical, carry a multiplicative factor of $\frac{1}{2}$.

Idea of the proof

Let $Y_t := \mathbb{E}_t\left[A_T\right]$. Then Y is a martingale. As in (3) write

$$\Lambda_t^T(\epsilon) := \log \mathbb{E}_t\left[e^{\epsilon A_T}\right] = \sum_{n=1}^{\infty} \epsilon^n\, \mathbb{K}_t^n(T).$$

Since $\Lambda_T^T(\epsilon) = 0$ and $\mathbb{E}_t e^{\epsilon Y_{t,T}} = e^{\Lambda_t^T(\epsilon)}$,

$$\mathbb{E}_t e^{\epsilon Y_T + \Lambda_T^T(\epsilon)} = e^{\epsilon Y_t + \Lambda_t^T(\epsilon)},$$

which exhibits $(e^{\epsilon Y_t + \Lambda_t^T(\epsilon)} : 0 \leq t \leq T)$, for fixed T, as an exponential martingale.

By Itô's Formula, $L^\epsilon := \epsilon Y + \Lambda^T(\epsilon)$ is a stochastic logarithm. Then $L^\epsilon + \frac{1}{2}\langle L^\epsilon \rangle$ is a martingale on $[0, T]$ and

$$\mathbb{E}_t \left[\epsilon Y_{t,T} - \Lambda_t^T(\epsilon) + \tfrac{1}{2}\langle \epsilon Y + \Lambda_t^T(\epsilon) \rangle_{t,T} \right] = 0 \; .$$

Insert $\Lambda_t^T(\epsilon) = \epsilon^2 \mathbb{K}_t^2 + \epsilon^3 \mathbb{K}_t^3 + \ldots$, and collect terms of order $[\epsilon^n]$, setting them to zero.

$[\epsilon^1]$: $\mathbb{E}_t \left[Y_{t,T} \right] = 0$.

$[\epsilon^2]$: $\mathbb{K}_t^2 = \frac{1}{2}\mathbb{E}_t \langle Y \rangle_{t,T} = \frac{1}{2}(Y \diamond Y)_t(T)$.

$[\epsilon^3]$: $\mathbb{K}_t^3 = \mathbb{E}_t\langle Y, \mathbb{K}^2\rangle_{t,T} = (Y \diamond \mathbb{K}^2)_t(T)$.

$[\epsilon^4]$:
$$\begin{aligned} \mathbb{K}_t^4 &= \mathbb{E}_t\langle Y, \mathbb{K}^3\rangle_{t,T} + \tfrac{1}{2}\mathbb{E}_t\langle \mathbb{K}^2, \mathbb{K}^2\rangle_{t,T} \\ &= (Y \diamond \mathbb{K}^3)_t(T) + \tfrac{1}{2}(\mathbb{K}^2 \diamond \mathbb{K}^2)_t(T). \end{aligned}$$

Setting $\mathbb{K}^1 := Y$, the general term is given by

$$\mathbb{K}_t^{n+1}(T) = \frac{1}{2}\sum_{k=1}^{n}(\mathbb{K}^k \diamond \mathbb{K}^{n+1-k})_t(T).$$

Example: $\mathbb{K}^3$ and the third cumulant

For higher n, the forest expansion encodes relations that are increasingly complex to derive by hand. For example:

$$
\begin{aligned}
(Y \diamond (Y \diamond Y))_t(T) &= Y \diamond (\mathbb{E}_\bullet \langle Y \rangle_T - \langle Y \rangle_\bullet) = Y \diamond (\mathbb{E}_\bullet \langle Y \rangle_T) \\
&= \mathrm{cov}_t(Y_T, \langle Y \rangle_T) = \mathrm{cov}_t(Y_{t,T}, \langle Y \rangle_{t,T}) \,.
\end{aligned}
$$

From basic properties of Hermite polynomials

$$
H_3(Y_{t,T}, \langle Y \rangle_{t,T}) = Y_{t,T}^3 - 3 Y_{t,T} \langle Y \rangle_{t,T}
$$

is a martingale increment, with zero (t-conditional) expectation. Thus

$$
\mathbb{K}_t^3(T) = \tfrac{1}{2} (Y \diamond (Y \diamond Y))_t(T) = \frac{1}{3!} \mathbb{E}_t \left[Y_{t,T}^3 \right] .
$$

Application: MGF of the Lévy area

Theorem (P. Lévy)

Let $\{X, Y\}$ be 2-dimensional standard Brownian motion, and stochastic ("Lévy") area be given by

$$\mathcal{A}_t = \int_0^t (X_s\, dY_s - Y_s\, dX_s) \ .$$

Then, for $T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\mathbb{E}_0\left[e^{\mathcal{A}_T}\right] = \frac{1}{\cos T} \ .$$

- In particular, we will see how to compute trees in practice.

First term

First,

$$\mathbb{K}^2 = \frac{1}{2}$$

$$\begin{aligned} &= \frac{1}{2}(\mathcal{A} \diamond \mathcal{A})_t(T) \\ &= \frac{1}{2}\int_t^T \left(\mathbb{E}_t\left[X_s^2\right] + \mathbb{E}_t\left[Y_s^2\right]\right)\, ds \\ &= \frac{1}{2}(T-t)^2 + \frac{1}{2}(X_t^2 + Y_t^2)(T-t). \end{aligned}$$

In particular,

$$d\mathbb{K}_s^2 = (X_s\, dX_s + Y_s\, dY_s)(T-s) + \mathrm{BV},$$

where BV denotes a bounded variation term.

- Note that BV terms do not contribute to diamond trees.

Second term

Similarly, recalling that $d\mathbb{K}^1_s = X_s\, dY_s - Y_s\, dX_s$,

$$\mathbb{K}^3 = \mathbb{K}^1 \diamond \mathbb{K}^2 =$$

$$\begin{aligned} &= \mathbb{E}_t\left[\int_t^T d\langle \mathbb{K}^1, \mathbb{K}^2\rangle_s\right] \\ &= \mathbb{E}_t\left[\int_t^T [XY\, d\langle Y\rangle_s - YX\, d\langle X\rangle_s]\,(T-s)\right] = 0. \end{aligned}$$

- It is easy to check that all odd forests vanish.

$\mathbb{K}^4$

$$
\begin{aligned}
\mathbb{K}^4 &= \frac{1}{2}\,\mathbb{K}^2 \diamond \mathbb{K}^2 = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\,\mathbb{E}_t\left[\int_t^T \left[X_s^2\,d\langle X\rangle_s + Y_s^2\,d\langle Y\rangle_s\right](T-s)^2\right] \\
&= \frac{1}{2}\int_t^T \left(\mathbb{E}_t\left[X_s^2\right] + \mathbb{E}_t\left[Y_s^2\right]\right)(T-s)^2\,ds \\
&= \int_t^T (s-t)\,(T-s)^2\,ds + \frac{1}{2}\left(X_t^2 + Y_t^2\right)\int_t^T (T-s)^2\,ds \\
&= \frac{1}{12}(T-t)^4 + \frac{1}{2}\left(X_t^2 + Y_t^2\right)\frac{1}{3}(T-t)^3.
\end{aligned}
$$

- It is now clear how to extend this computation to all orders.

The general pattern

We see that for each even n, $\mathbb{K}_t^n(T) = a_n\, I_t^{(n)}(T)$ for some $a_n \in \mathbb{Q}$ where

$$\begin{aligned} I_t^{(n)}(T) &= \frac{1}{2}\int_t^T \left(\mathbb{E}_t\left[X_s^2\right] + \mathbb{E}_t\left[Y_s^2\right]\right)(T-s)^{n-2}\, ds \\ &= \frac{(T-t)^n}{n(n-1)} + \frac{1}{2}\left(X_t^2 + Y_t^2\right)\frac{1}{n-1}(T-t)^{n-1}. \end{aligned}$$

To compute the forests $\mathbb{K}^n$, we need the following lemma.

Lemma

$$\left(I^{(m)} \diamond I^{(n)}\right)_t(T) = \frac{2}{(m-1)(n-1)}\, I_t^{(n+m)}(T).$$

More terms

- Note from above that $\mathbb{K}^2 = I^{(2)}$ and $\mathbb{K}^4 = I^{(4)}$.
- Applying the lemma

$$\begin{aligned}\mathbb{K}^6 &= I^{(4)} \diamond I^{(2)} = \frac{2}{3 \cdot 1}\, I^{(6)} \\ &= \frac{(T-t)^6}{45} + \frac{2}{3}\,\frac{1}{2}\,(X_t^2 + Y_t^2)\,\frac{1}{5}\,(T-t)^5.\end{aligned}$$

- In principle, we could go on for ever, computing forests (or cumulants) in this way.
    - As we show in [FGR20], without much extra effort, we can sum all these cumulants and so recover Lévy's theorem.

Remark

As a comparison, Levin and Wildon[LW08] obtain Lévy's theorem from (a much harder) moment expansion.

Breaking exponential martingales

Consider a martingale $(a\,Y_t)$ with stochastic exponential

$$\mathcal{E}(aY)_T = \exp\left\{ a\,Y_T - \frac{a^2}{2}\langle Y\rangle_T \right\}.$$

Then

$$\mathbb{E}_t\left[e^{a\,Y_T - \frac{a^2}{2}\langle Y\rangle_{t,T}} \right] = e^{a\,Y_t} \tag{4}$$

with "trivial" right-hand side.

- The individual cumulants have more structure than their sum (given by $a\,Y_t$).

- Applying the $\mathbb{K}$-recursion to the LHS of (4) gives

$$\mathbb{K}^1 = aY_t - \frac{a^2}{2}\mathbb{E}_t\left[\langle Y\rangle_{t,T}\right], \quad \mathbb{K}^2 = \frac{1}{2}\left(aY - \frac{a^2}{2}\langle Y\rangle\right)^{\diamond 2}, \ldots$$

  and all terms homogenous in $a^k, k \geqslant 2$, cancel upon summation of (finitely many) trees.

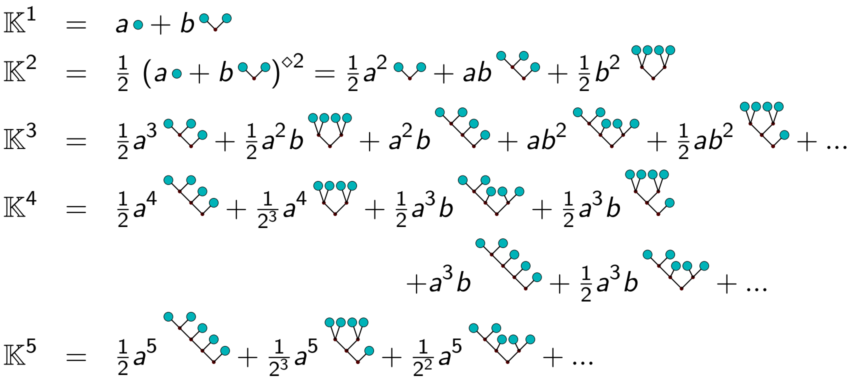
- The root cause is that (with $b = -a^2/2$)

$$\mathbb{K}^1 = aY_t + b\,\mathbb{E}_t\langle Y\rangle_{t,T} = a$$  $$+\, b$$  , (5)

  is a linear combination of trees with different number of leaves, and this propagates to all further terms in the $\mathbb{K}$-expansion.

The first few forests

In fact, applying the $\mathbb{K}$-recursion (1) with $\mathbb{K}^1$ given by (5), for arbitrary a, b, and neglecting trees with 6 or more leaves, the first few $\mathbb{K}$-forests are given by



Forest reordering

- We can choose to reorder the $\mathbb{K}$-forest series into forests of trees grouped by number of leaves.
- Define $\mathbb{F}^\ell$ to be the (finite) linear combination of trees in the $\mathbb{K}$-expansion with $\ell \geqslant 1$ leaves.
  - Since the corresponding $\mathbb{F}$-expansion will be just a reordered version of the $\mathbb{K}$-expansion, it inherits the convergence properties of the cumulant expansion given in Lemma 2.
- Since $a Y_t$ is the only tree with one leaf, $\mathbb{F}^1 = a\,Y = a\,\bullet$.
- Then

$$\sum_{k \geqslant 1} \mathbb{K}^k = a\,\bullet + \sum_{\ell \geqslant 2} \mathbb{F}^\ell.$$

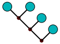
Graphical reordering

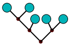
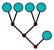
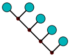
Reordering the $\mathbb{K}$-forests according to number of leaves, we see that the first few $\mathbb{F}$-forests are given by

$$\mathbb{F}^1 = a$$ 

$$\mathbb{F}^2 = \left(\tfrac{1}{2}a^2 + b\right)$$ 

$$\mathbb{F}^3 = a\left(\tfrac{1}{2}a^2 + b\right)$$ 

$$\mathbb{F}^4 = \tfrac{1}{2}\left(\tfrac{1}{2}a^2 + b\right)^2$$  $$+\, a^2\left(\tfrac{1}{2}a^2 + b\right)$$ 

$$\mathbb{F}^5 = a\left(\tfrac{1}{2}a^2 + b\right)^2$$  $$+\, \tfrac{1}{2}a\left(\tfrac{1}{2}a^2 + b\right)^2$$  $$+\, a^3\left(\tfrac{1}{2}a^2 + b\right)$$  .

- Note in particular that the $\mathbb{F}$ forests are simpler.

$\mathbb{F}$-recursion

The $\mathbb{F}$-forests satisfy the following recursion relation.

Theorem 4

With $\mathbb{F}^2 = \left(\frac{1}{2}a^2 + b\right)$  *and* $\forall k > 2$,

$$\mathbb{F}^k = \frac{1}{2}\sum_{j=2}^{k-2} \mathbb{F}^{k-j} \diamond \mathbb{F}^j + \left(a\, Y \diamond \mathbb{F}^{k-1}\right), \tag{6}$$

and we have, for sufficiently small a *and* b,

$$\mathbb{E}_t\left[e^{a Y_T + b \langle Y \rangle_{t,T}}\right] = e^{a\, Y_t + \sum_{\ell \geq 2} \mathbb{F}^\ell}. \tag{7}$$

Breaking the exponential martingale

- In the exponential martingale case $b = -a^2/2$, all the $\mathbb{F}$ forests vanish and we retrieve the exponential martingale

$$\exp\left\{a Y_t - \frac{a^2}{2}\langle Y\rangle_t\right\}.$$

- In the case $b \neq -a^2/2$, (7) can be viewed as breaking the rigid exponential martingality condition $b = -a^2/2$.
- In this case, A_T arises from the process $A_t = a Y_t + b\langle Y\rangle_t$.
  - The $\mathbb{K}$-expansion is *a priori* indifferent to this additional structure.

Stochastic volatility

Now we return to the financial mathematics context that originally gave rise to our result.

- Let S be a strictly positive continuous martingale.
- Then $X := \log S$ is a semimartingale with quadratic variation process $\langle X \rangle$.
- Spot variance and forward variance are given by

$$\begin{aligned} v_t \, dt &:= d\langle X \rangle_t \\ \xi_t(T) &= \mathbb{E}_t \left[v_T \right]. \end{aligned}$$

- It is natural to specify a model in forward variance form.
  - Forward variances are tradable assets (unlike spot variance).
  - We get a family of martingales indexed by their individual time horizons T.

Triple joint MGF

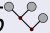
Theorem 5

For $a, b, c \in \mathbb{R}$ *sufficiently small we have, with* $\bar{b} = b - \frac{1}{2}\, a$,

$$
\begin{aligned}
&\mathbb{E}_t \left[e^{a\, X_T + b\, \langle X \rangle_{t,T} + c\, v_T} \right] \\
&= \exp \left\{ a\, X_t + \bar{b}\, (X \diamond X)_t(T) + c\, \xi_t(T) + \sum_{k=2}^{\infty} \mathbb{K}_t^k \right\}, \qquad (8)
\end{aligned}
$$

with

$\mathbb{K}^1 = a\, X + \bar{b}\,(X \diamond X) + c\, \xi = a\,($  $) + \bar{b}$  $+ c\,($  $)$

$\mathbb{K}^2 = \frac{1}{2}\Big(a^2$  $+ \bar{b}^2$  $+ c^2$  $\Big) + a\bar{b}$  $+ ac$  $+ \bar{b}c$ .

Proof.

This is a direct consequence of Theorem 1: The time-T quantity of interest is

$$A_T := a\,X_T + b\,\langle X\rangle_{t,T} + c\,v_T$$

and it suffices to compute (using that $X + \frac{1}{2}\langle X\rangle$ is martingale),

$$\mathbb{E}_t\left[A_T\right] = a\,X_t + (b - \tfrac{1}{2}\,a)\,(X \diamond X)_t(T) + c\,\xi_t(T)\,.$$

□

- We can get the joint MGF of any random variables of interest in the same way.
  - For example, VIX futures are martingales. So the joint MGF of SPX and VIX is in principle computable!

Theorem 4 from Theorem 5

Remark

Theorem 4 can be seen as a corollary of Theorem 5. Indeed putting $c = 0$ in (8), with $Y_s = X_s + \frac{1}{2}\langle X\rangle_{t,s}$,

$$
\begin{aligned}
\mathbb{E}_t\left[e^{a\,Y_T + b\,\langle Y\rangle_{t,T}}\right] &= \mathbb{E}_t\left[e^{aX_T + \left(b + \frac{1}{2}a\right)\langle X\rangle_{t,T}}\right] \\
&= \exp\left\{a\,X_t + b\,(X \diamond X)_t(T) + \sum_{k=2}^{\infty} \mathbb{K}_t^k\right\} \\
&= \exp\left\{a\,Y_t + \sum_{\ell\geq 2} \mathbb{F}_t^\ell\right\},
\end{aligned}
$$

where the $\mathbb{F}^\ell$ satisfy the recursion (6).

Trees with colored leaves

- In Theorem 5 we wrote

$$\mathbb{K}^1 = a\,X + \bar{b}\,(X \diamond X) + c\,\xi = a\,({\color{gray}\bullet}) + \bar{b}\,{\color{gray}\bullet}\!\vee\!{\color{gray}\bullet} + c\,({\color{magenta}\bullet}).$$

- We could define $(X \diamond X) = M$, or ${\color{gray}\bullet}\!\vee\!{\color{gray}\bullet} = {\color{orange}\bullet}$, resulting in trees with leaves of three different colors.
  - X_t represents the log-stock price and $M_t(T)$ the expected total variance $\int_t^T \xi_t(u)\,du$.
- Then

$$\begin{aligned}
\mathbb{K}^1 &= a\,X + \bar{b}\,M + c\,\xi = a\,({\color{gray}\bullet}) + \bar{b}\,({\color{orange}\bullet}) + c\,({\color{magenta}\bullet}) \\
\mathbb{K}^2 &= \tfrac{1}{2}\left(a^2\,{\color{gray}\bullet}\!\vee\!{\color{gray}\bullet} + \bar{b}^2\,{\color{orange}\bullet}\!\vee\!{\color{orange}\bullet} + c^2\,{\color{magenta}\bullet}\!\vee\!{\color{magenta}\bullet}\right) + a\bar{b}\,{\color{gray}\bullet}\!\vee\!{\color{orange}\bullet} + ac\,{\color{gray}\bullet}\!\vee\!{\color{magenta}\bullet} + \bar{b}c\,{\color{orange}\bullet}\!\vee\!{\color{magenta}\bullet}.
\end{aligned}$$

- In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).

Recovering the exponentiation theorem of [AGR2020]

Setting $b = c = 0$ in Theorem 5 gives the following corollary:

Corollary

For sufficiently small $a \in \mathbb{R}$,

$$\log \mathbb{E}_t\left[e^{a X_T}\right] = \sum_{k=1}^{\infty} \mathbb{K}_t^k = a\,X_t + \tfrac{1}{2}a\,(a-1)\,M_t(T) + \sum_{\ell=3}^{\infty} \mathbb{F}_t^{\ell}, \quad (9)$$

where the $\mathbb{K}^k$'s are given by (1), starting with

$\mathbb{K}^1 = a\,X - \frac{1}{2}a\,(X \diamond X) = a\,\bullet - \frac{1}{2}a$  .

On the other hand, Corollary 3.1 of [AGR2020] reads:

Corollary

The cumulant generating function (CGF) is given by

$$\psi_t^T(a) = \log \mathbb{E}_t\left[e^{i\,a\,X_t}\right] = i\,a\,X_t - \frac{1}{2}a\,(a+i)\,M_t(T) + \sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a). \tag{10}$$

where the $\tilde{\mathbb{F}}_k$ satisfy the recursion

$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+\mathrm{i})\,M_t = -\frac{1}{2}a(a+\mathrm{i})\bullet$ and for $k>0$,

$$\tilde{\mathbb{F}}_k = \frac{1}{2}\sum_{j=0}^{k-2}\left(\tilde{\mathbb{F}}_{k-2-j}\diamond\tilde{\mathbb{F}}_j\right) + \mathrm{i}a\left(X\diamond\tilde{\mathbb{F}}_{k-1}\right). \tag{11}$$

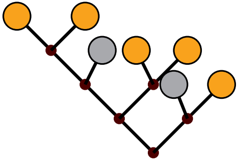
- With the identification $\tilde{\mathbb{F}}_k = \mathbb{F}^{k+2}$, formulae (9) and (10), and the recursions (6) and (11) are equivalent.

Simple diamond rules

- For $k > 0$, the kth forest $\tilde{\mathbb{F}}_k$ contains all trees with $k+2$ leaves where X is counted as a single leaf, and M as a double leaf.
  - Now we see why!
- Prefactor computation:
  - Work from the bottom up.
  - If child subtrees immediately below a diamond node are identical, carry a multiplicative factor of $\frac{1}{2}$.

Example: One tree in $\tilde{\mathbb{F}}_{10}$

$$\frac{1}{4}(X \diamond M) \diamond ((M \diamond M) \diamond (X \diamond (M \diamond M)))$$



- The factor $\frac{1}{4}$ relates to two symmetric sub-trees.

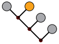
Applying the recursion (11), the first few $\tilde{\mathbb{F}}$ forests are given by

$$\tilde{\mathbb{F}}_0 = -\tfrac{1}{2}a(a+\mathrm{i})$$  $$= -\tfrac{1}{2}a(a+\mathrm{i})$$ 

$$\tilde{\mathbb{F}}_1 = -\tfrac{\mathrm{i}}{2}a^2(a+\mathrm{i})$$ 

$$\tilde{\mathbb{F}}_2 = \tfrac{1}{2^3}a^2\,(a+\mathrm{i})^2$$  $$+\tfrac{1}{2}a^3\,(a+\mathrm{i})$$ 

$$\tilde{\mathbb{F}}_3 = (\tilde{\mathbb{F}}_0 \diamond \tilde{\mathbb{F}}_1) + \mathrm{i}a$$  $$\diamond\, \tilde{\mathbb{F}}_2$$

$$= \tfrac{\mathrm{i}}{2^2}a^3\,(a+\mathrm{i})^2$$  $$+\tfrac{\mathrm{i}}{2^3}a^3\,(a+\mathrm{i})^2$$  $$+\tfrac{\mathrm{i}}{2}a^4\,(a+\mathrm{i})$$  .

- Note that the total probability and martingale constraints are satisfied for each tree.
  - That is $\psi_t^T(0) = \psi_t^T(-\mathrm{i}) = 0$.

Variance and gamma swaps

The variance swap is given by the fair value of the log-strip:

$$\mathbb{E}_t\left[X_T\right] = (-\mathrm{i})\,\psi_t^{T\prime}(0) = X_t - \frac{1}{2}\,M_t(T)$$

and the gamma swap (wlog set $X_t = 0$) by

$$\mathbb{E}_t\left[X_T\,e^{X_T}\right] = -\mathrm{i}\,\psi_t^{T\prime}(-\mathrm{i}).$$

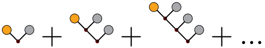
Remark

The point is that we can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.

The gamma swap

It is easy to see that only trees containing a single • leaf will survive in the sum after differentiation when $a = -\mathrm{i}$ so that

$$\sum_{k=1}^{\infty} \tilde{\mathbb{F}}'_k(-\mathrm{i}) = \frac{\mathrm{i}}{2} \sum_{k=1}^{\infty} X^{\diamond k} M =$$



where $X^{\diamond k} M$ is defined recursively for $k > 0$ as $X^{\diamond k} M = X \diamond X^{\diamond(k-1)} M$. Then the fair value of a gamma swap is given by

$$\mathcal{G}_t(T) = 2\,\mathbb{E}_t\left[X_T\, e^{X_T}\right] = M_t(T) + \sum_{k=1}^{\infty} X^{\diamond k} M. \tag{12}$$

Remark

Equation (12) allows for explicit computation of the gamma swap for any model written in forward variance form.

The leverage swap

We deduce that the fair value of a leverage swap is given by

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - M_t(T) = \sum_{k=1}^{\infty} X^{\diamond k} M \tag{13}$$

- The leverage swap is expressed explicitly in terms of diamond products of the spot and vol. processes.
  - If spot and vol. processes are uncorrelated, the fair value of the leverage swap is zero.

An explicit expression for the leverage swap!

$\mathcal{L}_t(T)$ directly from the smile

- Let
$$d_\pm(k) = \frac{-k}{\sigma_{\rm BS}(k,T)\sqrt{T}} \pm \frac{\sigma_{\rm BS}(k,T)\sqrt{T}}{2}$$
and following Fukasawa, denote the inverse functions by $g_\pm(z) = d_\pm^{-1}(z)$. Further define
$$\sigma_\pm(z) = \sigma_{\rm BS}(g_\pm(z),T)\,\sqrt{T}.$$
- It is a well-known corollary of Matytsin's characteristic function representation in [Mat00], that
$$M_t(T) = \int dz\, N'(z)\,\sigma_-^2(z).$$
- The gamma swap is given by
$$\mathcal{G}_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_+^2(z).$$

Skewness

As is well-known, the first three central moments are easily computed from cumulants by differentiation. For example, skewness is given by

$$
\begin{aligned}
\mathcal{S}_t(T) &:= \mathbb{E}_t\left[(X_T - \bar{X}_T)^3\right] \\
&= (-\mathrm{i})^3\, \psi_t^{T\,'''}(0) \\
&= -\frac{3}{2}(M \diamond M)_t(T) - \frac{3}{8}(M \diamond (M \diamond M))_t(T) \\
&\quad + \frac{3}{2}(M \diamond (X \diamond M))_t(T) + 3\,(X \diamond M)_t(T) \\
&\quad + \frac{3}{4}(X \diamond (M \diamond M))_t(T) - 3\,(X \diamond (X \diamond M))_t(T).
\end{aligned} \tag{14}
$$

An explicit expression for skewness!

The Bergomi-Guyon smile expansion

- The Bergomi-Guyon (BG) smile expansion (Equation (14) of [BG12]) reads

$$\sigma_{\mathrm{BS}}(k,T) = \hat{\sigma}_T + \mathcal{S}_T\, k + \mathcal{C}_T\, k^2 + \mathcal{O}(\epsilon^3)$$

  where the coefficients $\hat{\sigma}_T$, $\mathcal{S}_T$ and $\mathcal{C}_T$ are complicated combinations of trees such as $X \diamond M$.
- As we have seen, such trees are formally easily computable in any stochastic volatility model written in forward variance form.
  - The beauty of the BG expansion is that it yields direct relationships between the smile and the explicit formulation of a model in forward variance form.

Bergomi-Guyon to higher order

- We can extend the Bergomi-Guyon expansion to any desired order using our formal expression for the CGF in terms of forests.
- To second order, ATM total variance is given by

$$\sigma^2_{\mathrm{BS}}(k,T)\,T = M_t(T) + \epsilon\, a_1(k) + \epsilon^2\, a_2(k) + \mathcal{O}(\epsilon^3) \qquad (15)$$

where, with $M_t(T) = w$ for ease of notation,

$$\begin{aligned} a_1(k) &= \left(\frac{k}{w} + \frac{1}{2}\right)(X \diamond M) \\ a_2(k) &= \frac{1}{4}(X \diamond M)^2 \left\{-\frac{5k^2}{w^3} - \frac{2k}{w^2} + \frac{3}{w^2} + \frac{1}{4w}\right\} \\ &\quad + \frac{1}{4}(M \diamond M)\left\{\frac{k^2}{w^2} - \frac{1}{w} - \frac{1}{4}\right\} \\ &\quad + (X \diamond (X \diamond M))\left\{\frac{k^2}{w^2} + \frac{k}{w} - \frac{1}{w} + \frac{1}{4}\right\}. \end{aligned}$$

Skewness, leverage, stochasticity and the volatility skew

- The explicit expression (14) for skewness applies to any stochastic volatility model written in forward variance form.
- There are numerous references in the literature to the connection between the implied volatility skew and both the skewness and the leverage swap.
- Our explicit expression shows how these three quantities are related.
  - Denoting the ATM implied volatility skew by $\psi_t(T)$, we have from the BG expansion (15) that to lowest order,

$$\psi_t(T) = \sqrt{\frac{w}{T}}\,\frac{1}{2\,w^2}\,(X \diamond M)_t(T)$$

and to lowest order in the forest expansion,

$$\frac{1}{3}\,\mathcal{S}_t(T) = (X \diamond M)_t(T) = \mathcal{L}_t(T).$$

Affine forward variance models

Following [GKR19] consider *forward variance models* of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\ d\xi_t(u) &= \kappa(u-t)\, \sqrt{v_t}\, dW_t, \end{aligned}$$

with $d\langle W, Z\rangle_t = \rho\, dt$.

- This class of models includes classical and rough Heston.

Affine trees

In [FGR20], we prove the following lemma.

Lemma 6

In an affine forward variance model, all diamond trees take the form

$$\int_t^T \xi_t(u)\, h(T-u)\, du$$

for some function h.

Classical Heston

Example (Classical Heston)

In this case,

$$d\xi_t(u) = \nu\, e^{-\lambda\,(u-t)}\,\sqrt{v_t}\, dW_t.$$

Then, for example,

$$= (X \diamond M)_t(T) \;=\; \frac{\rho\,\nu}{\lambda} \int_t^T \xi_t(u)\,\left[1 - e^{-\lambda(T-u)}\right]\, du.$$

Rough Heston

Example (Rough Heston)

In this case, with $\alpha = H + 1/2 \in (1/2, 1)$ *(and with* $\lambda = 0$*),*

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)}\,(u-t)^{\alpha-1}\,\sqrt{v_t}\,dW_t.$$

Then, for example,

$$= M_t(T) = (X \diamond X)_t(T) = \int_t^T \xi_t(u)\,du,$$

$$= \frac{\nu^2}{\Gamma(\alpha)^2}\int_t^T \xi_t(u)\,du\left(\int_u^T (s-u)^{\alpha-1}\,ds\right)^2$$

$$= \frac{\nu^2}{\Gamma(1+\alpha)^2}\int_t^T \xi_t(u)\,(T-u)^{2\,\alpha}\,du.$$

- For a bounded forward variance curve ξ one then sees that diamond trees with k leaves are of order $(T-t)^{1+(k-2)\alpha}$.
- In this case, the $\mathbb{F}$-expansion (forest reordering according to number of leaves) has the interpretation of a short-time expansion, the concrete powers of which depend on the roughness parameter $\alpha = H + 1/2 \in (1/2, 1)$, cf. [CGP18, GR19].

The triple joint MGF of affine forward variance models

- Lemma 6 combined with Theorem 5 characterize the triple-joint MGF of X_T, $\langle X \rangle_T$ and v_T for an affine forward variance model.
  - Compare with Theorem 4.3 of [AJLP2019] and Proposition 4.6 of [GKR19].
- We obtain the convolutional form

$$\mathbb{E}_t\left[e^{a\,X_T+b\,\langle X \rangle_{t,T}+c\,v_T}\right] = \exp\left\{a\,X_t + (\xi \star g)(\tau;a,b,c)_t(T)\right\}\ .$$

- This is consistent with (and generalizes) Theorem 2.6 of [GKR19] where the same convolution Riccati equation appears, but with $g = g(\tau;a)$ instead of $(\tau;a,b,c)$ and different boundary conditions.

Computation of trees under rough Heston

Apart from from bounded variation terms (abbreviated as 'BV'), we have

$$\begin{aligned} dX_t &= \sqrt{v_t}\, dZ_t + \text{BV} \\ dM_t &= \int_t^T d\xi_t(u)\, du \\ &= \frac{\nu}{\Gamma(\alpha)} \sqrt{v_t} \left(\int_t^T \frac{du}{(u-t)^\gamma} \right) dW_t \\ &= \frac{\nu\, (T-t)^\alpha}{\Gamma(1+\alpha)} \sqrt{v_t}\, dW_t. \end{aligned}$$

The first order forest

There is only one tree in the forest $\tilde{\mathbb{F}}_1$.

$$\begin{aligned}
\tilde{\mathbb{F}}_1 = (X \diamond M)_t(T) &= \mathbb{E}_t\left[\int_t^T d\langle X, M\rangle_s\right] \\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}_t\left[\int_t^T v_s\,(T-s)^\alpha\,ds\right] \\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\int_t^T \xi_t(s)\,(T-s)^\alpha\,ds.
\end{aligned}$$

Higher order forests

Define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds\, \xi_t(s)\, (T-s)^{j\,\alpha}.$$

Then

$$\begin{aligned} dI_s^{(j)}(T) &= \int_s^T du\, d\xi_s(u)\, (T-u)^{j\,\alpha} + \mathrm{BV} \\ &= \frac{\nu\,\sqrt{v_s}}{\Gamma(\alpha)}\, dW_s \int_s^T \frac{(T-u)^{j\,\alpha}}{(u-s)^{\gamma}}\, du + \mathrm{BV} \\ &= \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\, \nu\,\sqrt{v_s}\,(T-s)^{(j+1)\,\alpha}\, dW_s + \mathrm{BV}. \end{aligned}$$

With this notation,

$$(X \diamond M)_t(T) = \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, I_t^{(1)}(T).$$

The second order forest

There are two trees in $\tilde{\mathbb{F}}_2$:

$$\begin{aligned}(M \diamond M)_t(T) &= \mathbb{E}_t\left[\int_t^T d\langle M, M\rangle_s\right] \\ &= \frac{\nu^2}{\Gamma(1+\alpha)^2}\int_t^T \xi_t(s)\,(T-s)^{2\,\alpha}\,ds \\ &= \frac{\nu^2}{\Gamma(1+\alpha)^2}\,I_t^{(2)}(T)\end{aligned}$$

and

$$\begin{aligned}(X \diamond (X \diamond M))_t\,(T) &= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}_t\left[\int_t^T d\langle X, I^{(1)}\rangle_s\right] \\ &= \frac{\rho^2\,\nu^2}{\Gamma(1+2\,\alpha)}\,I_t^{(2)}(T).\end{aligned}$$

The third order forest

Continuing to the forest $\tilde{\mathbb{F}}_3$, we have the following.

$$
\begin{aligned}
(M \diamond (X \diamond M))_t\,(T) &= \frac{\rho\,\nu^3}{\Gamma(1+\alpha)\,\Gamma(1+2\,\alpha)}\; I_t^{(3)}(T)\\
(X \diamond (X \diamond (X \diamond M)))_t\,(T) &= \frac{\rho^3\,\nu^3}{\Gamma(1+3\,\alpha)}\; I_t^{(3)}(T)\\
(X \diamond (M \diamond M))_t\,(T) &= \frac{\rho\,\nu^3\,\Gamma(1+2\,\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\,\alpha)}\; I_t^{(3)}(T).
\end{aligned}
$$

In particular, we easily identify the pattern

$$
\left(X^{\diamond k} M\right)_t (T) = \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)}\; I_t^{(k)}(T).
$$

The leverage swap under rough Heston

Using (13), we have

$$\begin{aligned}
\mathcal{L}_t(T) &= \sum_{k=1}^{\infty} X^{\diamond k} M_t(T) \\
&= \sum_{k=1}^{\infty} \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)} \int_t^T du\,\xi_t(u)\,(T-u)^{k\,\alpha} \\
&= \int_t^T du\,\xi_t(u)\,\{E_\alpha(\rho\,\nu\,(T-u)^\alpha)-1\}
\end{aligned}$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function.

An explicit expression for the leverage swap!

- Since we can impute the leverage swap $\mathcal{L}_t(t)$ from the smile for each expiration T, fast calibration of the rough Heston model is possible.

Summary

- We introduced the diamond product.
- We introduced the $\mathbb{K}$-expansion and sketched the proof.
  - We proved Lévy's Theorem as an example of its application.
- We showed how to reorder the $\mathbb{K}$-expansion to obtain the $\mathbb{F}$-expansion.
  - We obtained the Exponentiation Theorem of [AGR2020] as a corollary.
- We showed how easy computations can be in affine forward variance models.
  - Quick calibration of such models is one application.

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