Interest Rate and Credit Models

6. Convexity and CMS

Andrew Lesniewski

Baruch College
New York

Spring 2019

Outline

Convexity

- In financial lingo, *convexity* is a broadly understood and often non-specific term for nonlinear behavior of the price of an instrument as a function of evolving markets.
- Typically, such convexities reflect the presence of some sort of optionality embedded in the instrument.
- In this lecture we will focus on a number of convexities which arise in interest rates markets.
- Convex behavior in interest rate markets manifests itself as the necessity to include *convexity corrections* to various popular interest rates and they can be blessings and nightmares of market practitioners.
- From the perspective of financial modeling they arise as the results of valuation done under the "wrong" martingale measure.

Convexity

- Throughout this lecture we will be making careful notational distinction between stochastic processes, such as prices of zero coupon bonds, and their current (known) values.
- The latter will be indicated by the subscript 0. Thus, as in the previous lectures,
  (i) $P_0(t, T) = P(0, t, T)$ denotes the current value of the forward discount factor,
  (ii) $P(t, T) = P(t, t, T)$ denotes the time t value of the stochastic process describing the price of the zero coupon bond maturing at T.

LIBOR in arrears

- The simplest instrument leading to such convexity correction is a *LIBOR in arrears* swap.
- Consider a swap on which LIBOR fixes and pays on the start date of the accrual period S, rather than on its end date T.
- The present value of such a LIBOR payment is then

$$P = P_0(0, S)\mathrm{E}^{\mathrm{Q}_S}[L(S, T)], \tag{1}$$

  where, as usual, Q_S denotes the S-forward measure.
- The expected value is clearly taken with respect to the wrong martingale measure!

LIBOR in arrears

- The "natural" measure is the T-forward measure. Applying Girsanov's theorem,

$$P_0(0,S)\mathrm{E}^{\mathrm{Q}_S}[L(S,T)] = P_0(0,T)\mathrm{E}^{\mathrm{Q}_T}\Big[\frac{L(S,T)}{P(S,T)}\Big].$$

- The LIBOR in arrears forward is thus given by:

$$\begin{aligned}\mathrm{E}^{\mathrm{Q}_S}[L(S,T)] &= \mathrm{E}^{\mathrm{Q}_T}\Big[L(S,T)\,\frac{P_0(S,T)}{P(S,T)}\Big]\\ &= \mathrm{E}^{\mathrm{Q}_T}[L(S,T)\,] + \mathrm{E}^{\mathrm{Q}_T}\Big[L(S,T)\,\Big(\frac{P_0(S,T)}{P(S,T)} - 1\Big)\Big].\end{aligned}$$

- The first term on the right hand side is simply the LIBOR forward, while the second term is the in arrears convexity correction, which we shall denote by $\Delta(S,T)$, i.e.,

$$\mathrm{E}^{\mathrm{Q}_S}[L(S,T)] = L_0(S,T) + \Delta(S,T).$$

LIBOR in arrears

- Let us evaluate this correction using the Black model, i.e.

$$L(S,T) = L_0(S,T)e^{\sigma W(S) - \frac{1}{2}\sigma^2 S}.$$

- Key to the calculation will be the fact that

$$\mathrm{E}[e^{aW(t)}] = e^{\frac{1}{2}a^2 t}. \tag{2}$$

- We have

$$P(S,T) = \frac{1}{1 + \delta F(S,T)},$$

where $F(S,T)$ is the OIS forward, and δ is the coverage factor for the period $[S,\ T]$.

LIBOR in arrears

- Therefore, using (2),

$$\mathrm{E}^{Q_T}\Big[\frac{L(S,T)}{P(S,T)}\Big]=L_0(S,T)\big(1-\delta B(S,T)\big)+\delta L_0(S,T)^2\ e^{\sigma^2 S},$$

  where $B(S,T)$ denotes the (deterministic) LIBOR / OIS spread.
- After simple algebra we find that

$$\begin{aligned}\Delta(S,T)&=\mathrm{E}^{Q_T}\Big[L(S,T)\ \frac{P_0(S,T)}{P(S,T)}\Big]-L_0(S,T)\\&=L_0(S,T)\ \frac{\delta L_0(S,T)}{1+\delta F_0(S,T)}\ (e^{\sigma^2 S}-1).\end{aligned}$$

LIBOR in arrears

- In summary,

$$\Delta(S,T) = L_0(S,T)\theta\left(e^{\sigma^2 S} - 1\right), \tag{3}$$

where

$$\theta = \frac{\delta L_0(S,T)}{1+\delta F_0(S,T)}.$$

- Expanding the exponential to the first order, one can write the more familiar form for the convexity correction:

$$\Delta(S,T) \simeq L_0(S,T)\theta\sigma^2 S. \tag{4}$$

- This calculation was done under our usual assumption of deterministic LIBOR / OIS spread.
- In reality, since that spread is stochastic, there is an additional (very small) contribution to the convexity correction.
- The calculation above is an archetype for all approximate convexity computations and we will see it again.

Eurodollar futures / FRAs convexity corrections

- Another example of a convexity correction is that between a Eurodollar future and a FRA. What is its financial origin?
- Consider an investor with a long position in a Eurodollar contract.
- A FRA does not have any intermediate cash flows, while Eurodollar futures are marked to market daily by the Merc, and the margin account is adjusted accordingly.
- The implication for the investor's P&L is that it is negatively correlated with the dynamics of interest rates.
- If rates go up, the price of the contract goes down, and the investor needs to add money into the margin account, rather than investing it at higher rates (opportunity loss for the investor).

Eurodollar futures / FRAs convexity corrections

- If rates go down, the contract's price goes up, and the investor withdraws money out of the margin account and invests at a lower rate (opportunity loss for the investor again).
- The investor should thus demand a discount on the contract's price in order to be compensated for these adverse characteristics of his position compare to being long a FRA.
- As a result, the LIBOR calculated from the price of a Eurodollar futures contract has to be higher than the corresponding LIBOR forward.
- A Eurodollar future is cash settled at maturity (rather than at the end of the accrual period). The investor should be compensated by a lower price.
- This effect is analogous to the payment delay we discussed in the context of LIBOR in arrears and is relatively small.

Eurodollar futures / FRAs convexity corrections

- Mathematically, because of the daily variation margin adjustment, the appropriate measure defining the Eurodollar future is the spot measure Q_0.
- Therefore, the implied LIBOR rate $L_0^{\text{fut}}(S, T)$ is

$$L_0^{\text{fut}}(S, T) = \mathrm{E}^{Q_0}[L(S, T)]. \tag{5}$$

- The ED / FRA convexity correction is thus given by

$$\begin{aligned}\Delta_{\text{ED / FRA}}(T, T) &= L_0^{\text{fut}}(S, T) - L_0(S, T) \\ &= \mathrm{E}^{Q_0}[L(S, T)] - \mathrm{E}^{Q_T}[L(S, T)].\end{aligned} \tag{6}$$

- In order to derive a workable numerical value for $\Delta_{\text{ED / FRA}}(T, T)$, it is best to use a term structure model and we will address this issue later.

CMS swaps

- The acronym CMS stands for *constant maturity swap*, and it refers to a future fixing of a swap rate.
- For example, it may refer to the 10 year swap rate which will set 2 years from now. As we will see later in this lecture, CMS rates are different from the corresponding forward swap rates.
- CMS rates provide a convenient alternative to LIBOR as a floating index rate, as they allow market participants to express their views on the future levels of *long term rates* (for example, the 10 year swap rate).
- There are a variety of CMS based instruments actively traded in the markets. The simplest of them are CMS swaps, and CMS caps and floors.
- Other commonly traded instruments are CMS spread options. Valuation of these instruments will be the subject of the bulk of this lecture.

CMS swaps

- A fixed for floating *CMS swap* is a periodic exchange of interest payments on a fixed notional principal in which the floating rate is indexed by a *reference swap rate* (say, the 10 year swap rate).
- More precisely, the contract is specified as follows:
  - (i) The fixed leg pays a fixed coupon, quarterly, on the act/360 basis.
  - (ii) The floating leg pays the 10 year[1] swap rate which fixes two business days before the start of each accrual period.
  - (iii) The payments on the floating leg are made quarterly on the act/360 basis and are made at the end of each accrual period.

[1] Or whatever the tenor has been agreed upon.

CMS swaps

- CMS swaps are, in fact, commonly structured as *LIBOR for CMS swap*, rather than fixed for floating swaps.
- In a LIBOR for CMS swap, the fixed leg is replaced by a LIBOR leg. Namely:
  - (i′) The LIBOR leg pays a floating coupon equal to the LIBOR rate plus a fixed spread, quarterly, on the act/360 basis.
- CMS swaps are used by corporates or investors seeking to maintain a constant asset or liability duration of their portfolios.
- They also provide a natural vehicle for market participants who wish to take a view on the shape of the swap curve.

CMS caps and floors

- A *CMS cap* is a basket of calls on a swap rate of fixed tenor (say, 10 years) structured in analogy to a LIBOR cap or floor.
- For example, a 5 year cap on 10 year CMS struck at K is a basket of CMS caplets each of which:
  - (i) pays $\max(10 \text{ year CMS rate} - K, 0)$ applied to the notional principal, where the CMS rate fixes two business days before the start of each accrual period;
  - (ii) the payments are quarterly on the act/360 basis, and are made at the end of each accrual period.
- The definition of a CMS floor is analogous.

CMS caps and floors

- CMS caps and floors are used by portfolio managers whose risk is to longer term interest rates.
- For example, mortgage servicing companies are naturally short the mortgage rate, as rallying interest rates usually lead to higher prepayments and declining volume of business.
- Since the mortgage rate is believed to be strongly correlated with the 10 year swap rate, one way to hedge this risk is to purchase low strike CMS floors.

CMS spread options

- A CMS *spread option* involves two CMS rates, and it is a European call or put on the spread between these rates.
- For example, a 1 year 2Y / 10Y CMS spread call struck at K pays upon exercise

$$\max(10 \text{ year CMS rate} - 2 \text{ year CMS rate} - K, 0),$$

  applied to the notional principal, where the CMS rates fix two business days before the option expiration day.
- CMS spread options provide a natural way to hedge or express the view on the future shape of the swap curve.

Valuation of CMS swaps and caps / floors

- Using a swap rate as the floating rate makes a transaction a bit more difficult to price than a usual LIBOR based swap.
- Let us start with a single period $[T_0, \ T_p]$ CMS swap (a *swaplet*) whose fixed leg pays coupon C.
- Clearly, the PV of the fixed leg is

$$P^{\text{fix}} = C\delta \, P_0(0, T_p), \tag{7}$$

where δ is the coverage factor for the period $[T_0, \ T_p]$.

- The PV of the floating leg of the swaplet is

$$P^{\text{float}} = P_0(0, T_p)\delta \, \mathrm{E}^{\mathrm{Q}_{T_p}}[S(T_0, T)], \tag{8}$$

where Q_{T_p} denotes the T_p-forward martingale measure.

Valuation of CMS swaps and caps / floors

- Remember that T denotes the maturity of the reference swap starting on T_0[2], and not the end of the accrual period.
- As a consequence,

$$\begin{aligned} P^{\text{CMS swaplet}} &= P^{\text{fix}} - P^{\text{float}} \\ &= P_0(0, T_{\text{p}})\delta\, \mathrm{E}^{\mathrm{Q}_{T_{\text{p}}}}[C - S(T_0, T)]. \end{aligned} \tag{9}$$

- The present value of a CMS swap is obtained by summing up the contributions from all constituent swaplets.

[2] Say, the 10 year anniversary of T_0.

Valuation of CMS swaps and caps / floors

- The valuation of CMS caplets and floorlets is similar:

$$P^{\text{CMS caplet}} = P_0(0, T_p)\delta\, \mathsf{E}^{Q_{T_p}}[\max(S(T_0, T) - K, 0)] \ , \tag{10}$$

  and

$$P^{\text{CMS floorlet}} = P_0(0, T_p)\delta\, \mathsf{E}^{Q_{T_p}}[\max(K - S(T_0, T), 0)] \ . \tag{11}$$

- Not surprisingly, this implies a put / call parity relation for CMS: The PV of a CMS floorlet struck at K less the present value of a CMS caplet struck at the same K is equal to the present value of a CMS swaplet paying K.

Valuation of CMS swaps and caps / floors

- Let $C(T_0, T\,|\,T_\mathrm{p})$ denote the break even CMS rate, given by

$$C(T_0, T\,|\,T_\mathrm{p}) = \mathrm{E}^{\mathrm{Q}_{T_\mathrm{p}}}[S(T_0, T)]\ . \tag{12}$$

- The notation is a bit involved, so let us be very specific.
  - (i) T_0 denotes the start date of the reference swap (say, 1 year from now). This will also be the start of the accrual period of the swaplet.
  - (ii) T denotes the maturity date of the reference swap (say, 10 years from T_0).
  - (iii) T_p denotes the payment day on the swaplet (say, 3 months from T_0). This will also be the end of the accrual period of the swaplet.
- For the sake of completeness we should mention that one more date plays a role, namely the date on which the swap rate is fixed. This is usually two days before the start date, and we shall neglect its impact.

CMS rate

- The CMS rate is not a very intuitive concept! In this section we will express it in terms of more familiar quantities.
- Let $A(t, T_{\mathrm{val}}, T_0, T)$ denote the forward annuity function defined in Lecture Notes #4.
- For simplicity of notation, below we use the following abbreviations:

$$
\begin{aligned}
A_0(T_0, T) &= A(0, 0, T_0, T),\\
A_0(T_{\mathrm{val}}, T_0, T) &= A(0, T_{\mathrm{val}}, T_0, T),\\
A(T_0, T) &= A(t, t, T_0, T).
\end{aligned}
$$

- Let $C(T_0, T \,|\, T_{\mathrm{p}})$ denote the CMS rate given by (12).
- Our goal is to write (12) in a more intuitive form.

CMS rate

- First, we apply Girsanov's theorem in order to change from the measure Q_{T_p} to the measure Q associated with the annuity starting at T_0:

$$P_0(0, T_p)\mathrm{E}^{Q_{T_p}}[S(T_0, T)] = A_0(T_0, T)\mathrm{E}^{Q}\Big[\frac{S(T_0, T)P(T_0, T_p)}{A(T_0, T)}\Big].$$

- As a consequence,

$$\begin{aligned} C(T_0, T \,|\, T_p) &= \mathrm{E}^{Q_{T_p}}[S(T_0, T)] \\ &= \mathrm{E}^{Q}\Big[S(T_0, T)\frac{A_0(T_0, T_0, T)}{A(T_0, T)}\,\frac{P(T_0, T_p)}{P_0(T_0, T_p)}\Big]. \end{aligned} \tag{13}$$

- This formula looks awfully complicated! However, it has the advantage of being expressed in terms of the "natural" martingale measure.

CMS rate

- In order to interpret it, we write

$$\frac{A_0(T_0, T_0, T)}{A(T_0, T)} \frac{P(T_0, T_p)}{P_0(T_0, T_p)} = 1 + \Big(\frac{A_0(T_0, T_0, T)}{A(T_0, T)} \frac{P(T_0, T_p)}{P_0(T_0, T_p)} - 1\Big).$$

- Notice that, by the martingale property of the annuity measure,

$$\mathrm{E}^{\mathrm{Q}}[S(T_0, T)] = S_0(T_0, T),$$

the current value of the forward swap rate!

CMS rate

- As a result,

$$
\begin{aligned}
C(T_0, T \,|\, T_{\mathrm{p}}) &= S_0(T_0, T) + \mathrm{E}^{\mathrm{Q}}\Big[S(T_0, T)\Big(\frac{A_0(T_0, T_0, T)}{A(T_0, T)}\,\frac{P(T_0, T_{\mathrm{p}})}{P_0(T_0, T_{\mathrm{p}})} - 1\Big)\Big] \\
&= S_0(T_0, T) + \Delta(T_0, T \,|\, T_{\mathrm{p}}),
\end{aligned}
$$

  where $\Delta(T_0, T \,|\, T_{\mathrm{p}})$ denotes the *CMS convexity correction*, i.e. the difference between the forward swap rate and the CMS rate.

- We will argue that the CMS convexity correction can be attributed to two factors:
  (i) Intrinsics of the dynamics of the swap rate which we shall, somewhat misleadingly, delegate to the correlation effects between LIBOR and swap rate.
  (ii) Payment day delay.

CMS rate

- Correspondingly, we have the decomposition:

$$\Delta(T_0, T \,|\, T_{\mathrm{p}}) = \Delta_{\mathrm{corr}}(T_0, T \,|\, T_{\mathrm{p}}) + \Delta_{\mathrm{delay}}(T_0, T \,|\, T_{\mathrm{p}}). \tag{14}$$

- In order to see it, we substitute the identity

$$\begin{aligned}
&\frac{A_0(T_0, T_0, T)}{A(T_0, T)} \frac{P(T_0, T_{\mathrm{p}})}{P_0(T_0, T_{\mathrm{p}})} - 1 \\
&\quad = \Big(\frac{A_0(T_0, T_0, T)}{A(T_0, T)} - 1 \Big) + \frac{A_0(T_0, T_0, T)}{A(T_0, T)} \Big(\frac{P(T_0, T_{\mathrm{p}})}{P_0(T_0, T_{\mathrm{p}})} - 1 \Big).
\end{aligned}$$

into the representation (13) of the CMS convexity correction.

CMS rate

- This yields the expressions

$$\Delta_{\mathrm{corr}}(T_0, T) = \mathrm{E}^{\mathrm{Q}}\Big[S(T_0, T)\Big(\frac{A_0(T_0, T_0, T)}{A(T_0, T)} - 1\Big)\Big], \tag{15}$$

and

$$\Delta_{\mathrm{delay}}(T_0, T \,|\, T_{\mathrm{p}}) = \mathrm{E}^{\mathrm{Q}}\Big[S(T_0, T)\,\frac{A_0(T_0, T_0, T)}{A(T_0, T)}\Big(\frac{P(T_0, T_{\mathrm{p}})}{P_0(T_0, T_{\mathrm{p}})} - 1\Big)\Big]. \tag{16}$$

- Note that $\Delta_{\mathrm{delay}}(T_0, T \,|\, T_{\mathrm{p}})$ is zero, if the CMS rate is paid at the beginning of the accrual period.

Calculating the CMS convexity correction

- The formulas for the CMS convexity adjustments derived above are model independent, and one has to make choices in order to produce accurate values.
- The issue of accurate calculation of the CMS corrections is of great practical importance and has been the subject of intensive research.
- The difficulty lies, of course, in our ignorance about the details of the martingale measure Q.

Calculating the CMS convexity correction

- Among the proposed approaches we list the following three methods.
- *Black model style calculation*. This method is based on the assumption that the forward swap rate follows a lognormal process.
- *Replication method*. This method attempts to replicate the payoff of a CMS structure by means of European swaptions of various strikes, regardless of the nature of the underlying process.
- The replication method allows us to take the volatility smile effects into account by, say, using the SABR model.
- *Monte Carlo simulation in conjunction with a term structure model*[3]. This method is the most consistent of the three approaches, but it is somewhat slow and, additionally, its success depends on the accuracy of the term structure model.

[3] We shall discuss term structure models in the following lectures.

Calculating the CMS convexity correction

- We explain the first two methods as they lead to closed form results. The replication method is widely used in the industry.
- For tractability, both these methods require additional approximations.
- We assume that all day count fractions are equal to $1/f$, where f is the frequency of payments on the reference swap (typically, $f = 2$).
- Furthermore, we assume that all discounting is in terms of a single swap rate, namely the rate $S(t, T_0, T)$.
- In order to simplify the notation, we set $S(t) = S(t, T_0, T)$, $A(t) = A(t, T_0, T)$, etc.

Calculating the CMS convexity correction

- Within these approximations, the annuity function is given by[4]

$$
\begin{aligned}
A(t) &= \frac{1}{f}\sum_{j=1}^{n}\frac{1}{(1+S(t)/f)^{j}} \\
&= \frac{1}{S(t)}\Big(1-\frac{1}{(1+S(t)/f)^{n}}\Big).
\end{aligned}
\tag{17}
$$

- Similarly, the discount factor from the start date to the payment date is

$$
\begin{aligned}
P(t) &= \frac{1}{1+S(t)/f_{\mathrm{CMS}}} \\
&\simeq \frac{1}{(1+S(t)/f)^{f/f_{\mathrm{CMS}}}}\,,
\end{aligned}
\tag{18}
$$

where f_{CMS} is the frequency of payments on the CMS swap (typically $f_{\mathrm{CMS}} = 4$).

[4] Incidentally, this way of calculating the annuity function of a swap is adopted in some markets in the context of *cash settled swaptions*.

Black model method

- We are now ready to carry out the calculations.
- Let us Taylor expand $1/A(t)$ in powers of $S(t)$ around S_0:

$$
\begin{aligned}
\frac{1}{A(t)} &\simeq \frac{1}{A_0} + \frac{d}{dS_0}\Big(\frac{1}{A_0}\Big)(S(t) - S_0) \\
&= \frac{1}{A_0}\Big(1 + \frac{1}{S_0}\Big(1 - \frac{1}{1 + S_0/f}\,\frac{nS_0/f}{(1 + S_0/f)^n - 1}\Big)(S(t) - S_0)\Big) \\
&\equiv \frac{1}{A_0}\Big(1 + \theta_c\,\frac{S(t) - S_0}{S_0}\Big).
\end{aligned}
$$

- We assume that the swap rate follows a lognormal process, i.e.

$$
S(t) = S_0 e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}.
$$

Black model method

- Using (2) we find that

$$\mathrm{E}^{\mathrm{Q}}\Big[S\,\frac{A_0}{A}\Big] = S_0 + S_0\theta_c\big(e^{\sigma^2 T_0} - 1\big).$$

- Similarly,

$$\mathrm{E}^{\mathrm{Q}}\Big[S\,\frac{A_0}{A}\Big(\frac{P}{P_0} - 1\Big)\Big] \simeq -S_0\theta_d\big(e^{\sigma^2 T_0} - 1\big),$$

  where

$$\theta_d = \frac{S_0/f_{\mathrm{CMS}}}{1 + S_0/f}\,.$$

- Using (2), and reinstating the arguments we find the following expressions for the convexity corrections:

$$\begin{aligned}\Delta_{\mathrm{corr}}(T_0, T\,|\,T_{\mathrm{p}}) &\simeq S_0(T_0, T)\theta_c\big(e^{\sigma^2 T_0} - 1\big),\\ \Delta_{\mathrm{delay}}(T_0, T\,|\,T_{\mathrm{p}}) &\simeq -S_0(T_0, T)\theta_d\big(e^{\sigma^2 T_0} - 1\big).\end{aligned} \tag{19}$$

Black model method

- These are our approximate expressions for the CMS convexity corrections.
- Finally, we can combine the impact of correlations and payment delay into one formula,

$$\Delta(T, T \,|\, T_{\mathrm{p}}) \simeq S_0(T, T)(\theta_c - \theta_d)\big(e^{\sigma^2 T_0} - 1\big) \tag{20}$$

  where

$$\theta_c - \theta_d = 1 - \frac{S_0/f}{1 + S_0/f}\Big(\frac{f}{f_{\mathrm{CMS}}} + \frac{n}{(1 + S_0/f)^n - 1}\Big). \tag{21}$$

- This is the approximation to the CMS convexity correction derived in [1].
- Expanding the exponential, the convexity adjustment can also be written in the more traditional form:

$$\Delta(T_0, T \,|\, T_{\mathrm{p}}) \simeq S_0(T_0, T)(\theta_c - \theta_d)\sigma^2 T_0. \tag{22}$$

Replication method

- Before describing the replication method, we establish an identity which will play a key role in its development.
- The starting point is the first order Taylor theorem. Namely, for a twice continuously differentiable function $F(x)$,

$$F(x) = F(x_0) + F'(x_0)(x - x_0) + \int_{x_0}^{x} F''(u)(x-u)du. \tag{23}$$

- In order to facilitate the financial interpretation of this formula, we rewrite the remainder term as follows:

$$\int_{x_0}^{x} F''(u)(x-u)du = \int_{-\infty}^{x_0} F''(u)(u-x)^{+}du + \int_{x_0}^{\infty} F''(u)(x-u)^{+}du,$$

where, as usual, $x^{+} = \max(x, 0)$.

Replication method

- It is clear what we are after: the remainder term looks very much like a mixture of payoffs of calls and puts!
- In order to make this observation useful, we assume that we are given a diffusion process $X(t) \geq 0$, and use the formula above with $x = X(T)$[5]:

$$
\begin{aligned}
&F(X(T)) = F(X_0) + F'(X_0)(X(T) - X_0) + \\
&\int_0^{X_0} F''(K)(K - X(T))^+ dK + \int_{X_0}^{\infty} F''(K)(X(T) - K)^+ dK.
\end{aligned}
\tag{24}
$$

[5] Extension of the formula below to convex (rather than twice continuously differentiable) functions is a deep theorem, know as Tanaka's theorem.

Replication method

- Let now Q be a martingale measure such that $\mathrm{E}^{\mathrm{Q}}[X(t)] = X_0$. Then, taking expectations of both sides of the identity above yields:

$$\mathrm{E}^{\mathrm{Q}}[F(X(T))] = F(X_0) + \int_0^{X_0} F''(K)\, B^{\mathrm{put}}(T, K, X_0) dK + \int_{X_0}^{\infty} F''(K)\, B^{\mathrm{call}}(T, K, X_0) dK, \tag{25}$$

where

$$\begin{aligned} B^{\mathrm{call}}(T, K, X_0) &= \mathrm{E}^{\mathrm{Q}}[(X(T) - K)^+], \\ B^{\mathrm{put}}(T, K, X_0) &= \mathrm{E}^{\mathrm{Q}}[(K - X(T))^+]. \end{aligned} \tag{26}$$

- Formula (25) is the desired replication formula.
- It states that if $F(x)$ is the payoff of an instrument, its expected value at time T is given by its today's value plus the value of a basket of out of the money calls and puts weighted by the second derivative of the payoff evaluated at the strikes.

Replication method

- There is an alternative way of writing (25).
- We integrate by parts twice in (25), and note that (i) the boundary terms at 0 and ∞ vanish, and (ii) the following relations hold at X_0:

$$
\begin{aligned}
B^{\mathrm{put}}(T, X_0, X_0) - B^{\mathrm{call}}(T, X_0, X_0) &= 0, \\
\frac{\partial B^{\mathrm{put}}}{\partial K}(T, X_0, X_0) - \frac{\partial B^{\mathrm{call}}}{\partial K}(T, X_0, X_0) &= 1.
\end{aligned}
$$

Replication method

- As a result,

$$\mathrm{E}^{\mathrm{Q}}[F(X(T))] = \int_0^\infty F(K)\, g_T(K, X_0) dK, \tag{27}$$

where

$$\begin{aligned} g_T(K, X_0) &= \frac{\partial^2}{\partial K^2}\, B^{\text{call}}(T, K, X_0) \\ &= \frac{\partial^2}{\partial K^2}\, B^{\text{put}}(T, K, X_0) \\ &= \mathrm{E}^{\mathrm{Q}}[\delta(X(T) - K)]. \end{aligned} \tag{28}$$

- We already encountered $g_T(K, X_0)$ in Lecture Notes #4 and #5.
- It is the expected value of a security whose payoff at time T is given by Dirac's delta function (known as the *Arrow-Debreu security*), and is thus the implied terminal probability density of prices of the asset X at time T.

Replication method

- The replication method is more general than the Black model method, as it does not use any specific assumptions about the nature of the process for $S(t)$.
- It relies only on the fact that the time T expected value of any payoff function can be represented as today's value of the payoff (the "moneyness") plus the time value which is a the value of a suitable basket of calls and puts expiring at T.
- We shall now derive an explicit expression for the CMS convexity correction within the replication method.
- For simplicity, we discuss only the total CMS correction, but it is easy to extend the calculation to each of the components separately.

Replication method

- Note first that what the approximations in (17) and (18) amount to is that the Radon-Nikodym derivative in (13) is a function of one variable only, namely $S = S(T_{\text{start}})$.
- Specifically, let us denote the function on the right hand side of (17) by $I(S)$, and denote the function on the right hand side of (18) by $p(S)$.
- The Radon-Nikodym derivative, denoted by $R(S)$, can be written as

$$R(S) = \frac{I(S_0)p(S)}{I(S)\,p(S_0)}\ .$$

Replication method

- Then, (13) implies that

$$C(T_0, T \,|\, T_\mathrm{p}) \simeq \mathrm{E}^{\mathrm{Q}}[S(T_0, T)R(S(T_0, T))]. \tag{29}$$

- Define now $F(S) = SR(S)$, and observe that $F(S_0) = S_0$.
- Applying the replication formula (25) to this function, we arrive at the following approximate representation of the CMS rate:

$$\begin{aligned} C(T_0, T \,|\, T_\mathrm{p}) &\simeq S_0(T_0, T) + \\ &\int_0^{S_0} F''(K)\, B^{\mathrm{put}}(T, K, S_0)dK + \int_{S_0}^{\infty} F''(K)\, B^{\mathrm{call}}(T, K, S_0)dK. \end{aligned} \tag{30}$$

Replication method

- As a final step, we can rewrite the above expression in terms of receiver and payer swaptions.
- Recall from Lecture Note #5 that these are obtained, respectively, by multiplying $B^{\text{put}}(T, K, S_0)$ and $B^{\text{call}}(T, K, S_0)$ by the annuity function. Therefore,

$$\Delta(T, T \,|\, T_{\mathrm{p}}) \simeq \int_0^{S_0} \frac{F''(K)}{I(K)} \operatorname{Rec}(T, K, S_0) dK + \int_{S_0}^{\infty} \frac{F''(K)}{I(K)} \operatorname{Pay}(T, K, S_0) dK. \tag{31}$$

- This formula links directly the CMS convexity correction to the swaption market prices.
- In practice, it can be used in conjunction with the SABR volatility model. In this approach, the integral above is discretized, and each of the swaption prices is calculated based on the calibrated SABR model.

Replication method

- Consider a CMS caplet expiring T_0 years from now. Today's price of the caplet is

$$P^{\text{CMS caplet}} = P_0(0, T_{\text{p}}) \, \mathrm{E}^{\mathrm{Q}_{T_{\text{p}}}}[(S(T_0, T_0, T) - K)^+] \tag{32}$$

  (multiplied by the day count fraction and notional).
- Since the dynamics of the forward swap rate under the T_{p}-forward measure is not explicitly known we have to resort to approximations in order to compute the expected value above.
- Let $g_{T_0}(F, F_0)$ denote the terminal probability distribution density of the forward swap rate at expiry. Then

$$P = P_0(0, T_{\text{p}}) \int_{-\infty}^{\infty} (F - K)^+ g_{T_0}(F, F_0) dF. \tag{33}$$

Replication method

- The exact form of $g_{T_0}(F, F_0)$ is unknown. The two methods discussed above lead to somewhat different approximations to $g_{T_0}(F, F_0)$.
- Using the Black model approximation, $G_{T_0}(F, K)$ is simply the lognormal distribution corresponding to the process

$$\begin{aligned} dS(t) &= \sigma_{\text{CMS}}\, S(t)\, dW(t)\,, \\ S(0) &= C(T_0, T \,|\, T_{\text{p}}). \end{aligned} \tag{34}$$

- Here the CMS rate $C(T_0, T \,|\, T_{\text{p}})$ is calculated according to the method calculated above, and σ_{CMS} is the appropriate implied volatility.

Replication method

- Within the replication method, we construct the terminal probability distribution from the swaption market prices as discussed above.
- Note that, since the construction involves taking derivatives of the prices with respect to the strikes, it is important that we interpolate the prices smoothly as a function of the strike.
- We can, for example, use the interpolation given by the SABR model. Option prices are calculated by numerical computation of the integral (33).
- Prices of CMS floorlets are found in the same way.
- Prices of CMS caps and floors are calculated by taking sums of the constituent caplets or floorlets.

Triangle arbitrage

- The existence of options on CMS spreads means that there is potential for arbitrage between CMS rate options and CMS spread options.
- Such arbitrage may arise when the valuation model for spread options is inconsistent with the model used to price CMS options.
- To illustrate this phenomenon consider two CMS rates S_1 and S_2 (say a 2 year CMS rate and a 30 year CMS rate), and let $S_{12} = S_2 - S_1$ denote the spread between these rates.
- If K_1 denotes the strike on the option on S_1 and K_2 denotes the strike on S_2, then the following inequality must hold:

$$(S_2 - K_2)^+ - (S_1 - K_1)^+ \leq (S_{12} - K_{12})^+ \leq (S_1 - K_1)^+ + (S_2 - K_2)^+, \quad (35)$$

  where $K_{12} = K_2 - K_1$ is the difference of strikes. This follows from the fact that $(x + y)^+ \leq x^+ + y^+$.

Triangle arbitrage

- Taking expected values with respect to the appropriate measure, this translates into the following set of conditions:

$$
\begin{aligned}
\mathrm{E}[(S_2-K_2)^+]-\mathrm{E}[(S_1-K_1)^+] &\leq \mathrm{E}[(S_{12}-K_{12})^+]\\
&\leq \mathrm{E}[(S_1-K_1)^+]+\mathrm{E}[(S_2-K_2)^+],
\end{aligned}
\tag{36}
$$

- This in turn, after multiplying by the discount factor, provides bounds of the CMS spread option price in terms of CMS option prices.
- Violating these bounds implies existence of arbitrage between CMS options and spread options.
- In order to assure that CMS spread option prices obey these bounds, the industry has developed models based on the copula[6] methodology.
- A preferred approach is to use a well calibrated term structure model, as this approach guarantees to be arbitrage free.

[6] We will discuss copulas later in this course.

References

- Hagan, P.: CMS conundrums: pricing CMS swaps, caps, and floors, *Wilmott Magazine*, **March**, 38 - 44 (2004).
- McCloud, P.: The CMS triangle arbitrage, *Risk Magazine*, **January**, 126 - 131 (2011).
- Pelsser, A.: Mathematical theory of convexity correction, SSRN (2001).