

Interest Rate and Credit Models

14. Interest Rate Risk Management

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Outline

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- 3 Portfolio vega risk
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Risk metrics

- One of the most important tasks faced by a portfolio manager, a trader, or a risk manager is to manage the interest rate exposure of a portfolio of fixed income securities such as government bonds, corporate bonds, mortgage backed securities, structured interest rate products, etc.
- This interest rate risk may manifest itself in various ways:
 - (i) risk to the level of rates,
 - (ii) risk to the convexity of instruments, and
 - (iii) risk to the volatility of rates.

Risk metrics

- Traditional risk measures of options are the *greeks*: delta, gamma, vega, theta, etc.¹, see for example [6].
- Recall, for example, that the delta of an option is the derivative of the premium with respect the underlying.
- This poses a bit of a problem in the world of interest rate derivatives, as the interest rates play a dual role in the option valuation formulas:
 - (i) as the underlyings, and
 - (ii) as the discount rates.
- One has thus to differentiate both the underlying *and* the discount factor when calculating the delta of a swaption!

¹ Rho, vanna, volga,...

Risk metrics

- The key issue is to quantify this exposure and, if required, offset aspects it by taking positions in liquid vanilla instruments such as Eurodollar futures, swaps, swaptions, caps/floors, etc.
- In addition to the various facets of interest rate risk, fixed income portfolios carry other kinds of risk.
- For example, government bonds carry foreign exchange risk and sovereign credit risk, corporate bonds are exposed to credit and liquidity risk, and mortgage backed securities have prepayment and credit risk. CAT bonds carry risks to natural catastrophes such as earth quakes and hurricanes.
- These other types of risk are the defining characteristics of the relevant instruments and are, in fact, their *raison d'être*.

Risk metrics

- We discuss the idiosyncratic risk specific to mortgage backed securities in this class.
- Credit risk management is discussed in the next set of notes.
- There are other widely used risk metrics such as the value at risk (VaR) and stress tests, which we will discuss in this lecture.
- The focus by VaR is less myopic than the sensitivity measure given by the greeks; instead it is a measure of the global tail risk. Under a given level of risk tolerance, what is the maximum loss that a portfolio can sustain?
- VaR is a tool used by banks to set aside their economic capital, clearing houses to set the margin levels, and hedge funds to set the risk capital to their traders.

Delta

- We begin with the dominant portion of the interest rate risk, namely the delta risk. Traditionally, this risk has been designated to as the *duration risk*.
- We let Π denote this portfolio, whose detailed composition is not important for our discussion. We will discuss two commonly used approaches to measure the interest rate risk of Π :
 - (i) Sensitivity to the inputs
 - (ii) Sensitivity to the forward curve
- Two methods of computing the delta are commonly used in the industry.

Input perturbation sensitivities

- In this approach we compute the sensitivities of the portfolio to the benchmark instruments used in the curve construction:
 - (i) Compute the *partial DVO1s* of the portfolio Π to each of the benchmark instruments B_i : We shift each of the benchmark rates down 1 bp and calculate the corresponding changes $\delta_i \Pi$ in the present value of the portfolio.
 - (ii) Compute the DVO1s $\delta_i B_i$ of the present values of the benchmark instruments under these shifts.
 - (iii) The hedge ratios Δ_i of the portfolio to the benchmarks are given by:

$$\Delta_i = \frac{\delta_i \Pi}{\delta_i B_i} .$$

Input perturbation sensitivities

- We then replicate the risk of the portfolio by means of a portfolio consisting of the suitably weighted benchmark instruments.
- This way of computing portfolio risk has the disadvantage that the shifts of each of the individual inputs (while keeping the others fixed) into the (multi-)curve construction propagate erratically throughout the entire curve, leading thus to its unintuitive shapes.

Regression based sensitivities

- An alternative and more robust approach consists in computing the sensitivities of the portfolio to a number of virtual “micro scenarios”, and expressing these sensitivities in terms of the sensitivities of a suitably selected hedging portfolio.
- First, we select a hedging portfolio and the rates scenarios. The hedging portfolio consists of vanilla instruments such as spot or forward starting swaps, Eurodollar futures, and forward rate agreements.
- The choice of instruments in the hedging portfolio should be made judiciously, based on understanding of the nature of the portfolio and liquidity of the instruments intended as hedges.
- Typically, a fixed income portfolio shows a great deal of sensitivity to the short end of the curve, and it is a good idea to include the first two years worth of Eurodollar futures.
- We let

$$\Pi_{\text{hedge}} = \{B_1, \dots, B_n\}.$$

denote this hedging portfolio.

Regression based sensitivities

- We now let \mathcal{C}_0 denote the current snapshot of the LIBOR / OIS multi-curve to which we refer as the base scenario.
- A micro scenario is a perturbation of the base scenario in which a segment $a \leq t < b$ of both the instantaneous LIBOR and OIS rates are shifted in parallel by a prescribed amount.
- For example, a micro scenario could result from \mathcal{C}_0 by shifting the first 3 month segment down by 1 basis point.
- Choose a *complete* set of non-overlapping micro scenarios

$$\mathcal{C}_1, \dots, \mathcal{C}_p.$$

Regression based sensitivities

- What we mean by this is that
 - (i) the shifted segments $a_i \leq t < b_i$ and $a_j \leq t < b_j$ of distinct C_i and C_j do not overlap, and
 - (ii) the union of all $a_i \leq t < b_i, i = 1, \dots, p$ is $(0, T_{\max})$.
 - (iii) There are of course, countless ways of choosing a complete set of non-overlapping micro scenarios.
- Ideally, we would select a large number of scenarios corresponding to narrowly spaced shifted segments but this may be impractical because of computational budget constraints.
- A reasonable alternative is a choice in which the short end of the curve is covered with narrow shifted segments which become sparser as we move toward the back end of the curve.

Regression based sensitivities

- We then compute the sensitivities of the portfolio and the hedging portfolio under these curve shifts.
- The vector $\delta\Pi$ of portfolio's *sensitivities* under these micro scenarios is

$$\delta_i\Pi = \Pi(C_i) - \Pi(C_0), \quad i = 1, \dots, p, \quad (1)$$

where by $\Pi(C_i)$ we denote the value of the portfolio given the shifted forward curve C_i .

- The matrix δB of sensitivities of the hedging instruments to these scenarios is

$$\delta_i B_j = B_j(C_i) - B_j(C_0). \quad (2)$$

- In order to avoid accidental co-linearities between its rows or columns, we should always use more micro scenario than hedging instruments.

Regression based sensitivities

- Finally, we translate the risk of the portfolio to the vector of hedge ratios with respect to the instruments in the hedging portfolio. We do this by means of *ridge regression*.
- The vector Δ of *hedge ratios* is calculated by minimizing the following objective function:

$$\mathcal{L}(\Delta) = \frac{1}{2} \|\delta B \Delta - \delta \Pi\|^2 + \frac{1}{2} \lambda \|Q \Delta\|^2. \quad (3)$$

- Here, λ is an appropriately chosen small smoothness parameter (similar to the Tikhonov regularizer!), and Q is the smoothing operator (say, the identity matrix). Explicitly,

$$\Delta = \left((\delta B)^t \delta B + \lambda Q^t Q \right)^{-1} (\delta B)^t \delta \Pi,$$

where the superscript t denotes matrix transposition.

Regression based sensitivities

- One can think of the component Δ_j as the sensitivity of the portfolio to the hedging instrument B_j .
- This method of calculating portfolio sensitivities is called the *ridge regression* method. It is very robust, and allows one to view the portfolio risk in a flexible way.
- In addition, one should quantify the exposure of the portfolio to the LIBOR / OIS basis by performing suitable sensitivity analysis of the portfolio under perturbing the spread curve.
- LMM is ideally suited to implement this approach, as its dynamics traces the evolution of the entire forward curve. Specifically, we proceed as follows.

Regression based sensitivities

- We use the current forward curve C_0 as the initial condition for the Monte Carlo simulations based on LMM.
- Using these paths, we calculate the values of the portfolio Π as well as each of the hedging instruments B_j (the latter may not require using simulations).
- This way we calculate the values $\Pi(C_0)$ and $B_j(C_0)$ introduced above.
- Next, for each of the micro scenarios C_i , $i = 1, \dots, p$, we generate the same number of Monte Carlo paths using C_i as the initial condition.
- It is important that the paths in each scenario are generated using the same seed for the random number generator (or the same Sobol numbers); otherwise additional sampling noise will be introduced into the process.
- We use them to compute the perturbed values $\Pi(C_i)$ and, if need be, $B_j(C_i)$.

Gamma

- The gamma of a portfolio is a measure of the non-constancy of its delta under the evolving market.
- In the case of an individual European option, the gamma is defined as the second derivative of the option price with respect to the underlying.
- Such a definition is rather useless for a portfolio of complex fixed income securities, as it would amount to calculating a noisy, high dimensional matrix of second partial derivatives.

Gamma

- A more practical way to look at the gamma risk is to view it as the change in the portfolio delta under specified macro scenarios:

$$\Xi_0, \Xi_1, \dots, \Xi_r, \quad (4)$$

with Ξ_0 base scenario (no change in rates).

- These could be, for instance, the scenarios produced by several principal components of the curve covariance matrix, or by specified hypothetical market moves.
- For example, we could take:

$$\begin{aligned} \Xi_{+50} &: \text{all rates up 50 basis points,} \\ \Xi_{+25} &: \text{all rates up 25 basis points,} \\ \Xi_{-25} &: \text{all rates down 25 basis points,} \\ \Xi_{-50} &: \text{all rates down 50 basis points.} \end{aligned} \quad (5)$$

Gamma

- For each of the macro scenarios, we calculate the deltas

$$\Delta_1, \dots, \Delta_r, \tag{6}$$

as explained in the previous section.

- The quantities:

$$\begin{aligned} \Gamma_1 &= \Delta_1 - \Delta_0, \\ &\vdots \\ \Gamma_r &= \Delta_r - \Delta_0, \end{aligned} \tag{7}$$

are the portfolio gammas under the corresponding scenarios.

- For intermediate market moves, the portfolio gamma can be calculated by linearly interpolating gammas corresponding to the specified macro scenarios.

Vega

- In order to quantify the vega risk we have to first design appropriate volatility scenarios.
- Earlier in the course we explained how LMM stores its internal representation \mathcal{S} of the volatility surface. We construct volatility micro scenarios by accessing \mathcal{S} and shifting selected non-overlapping segments.
- Let us call these scenarios

$$\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_q, \tag{8}$$

with $\mathcal{S}_0 = \mathcal{S}$ being the base scenario.

- Next, we choose a hedging portfolio Π_{hedge} which may consist of liquid instruments such as swaptions, caps and floors, Eurodollar options, or other instruments.

Vega

- The rest is a *verbatim* repeat of the delta story:
 - (i) We calculate the sensitivities of the portfolio to the volatility scenarios (8).
 - (ii) We calculate the sensitivities of the hedging portfolio to the volatility scenarios.
- Finally, we use ridge regression to find the hedge ratios.
- This method of managing the vega risk works well and allows one, in particular, to separate the exposure to swaptions from the exposure to caps / floors.
- The method outlined above allows us to measure and manage the at the money vega risk only.
- Proper smile risk management is done within the framework of stochastic volatility such as the SABR model.

Risk management under SABR

- The discussion above is general in the sense that no specific dynamic model has been assumed.
- The methods explained work for any generic interest rate model such as the Black, Hull-White, or LMM models.
- We now discuss the aspects of risk managements which are inherent to the SABR model and, by extension, to the SABR / LMM model.
- As explained above, the key measures of risk of a fixed income portfolio are sensitivities to selected segment of the curve.
- They can be calculated either by perturbing the inputs to the curve construction or by perturbing a segment of the OIS / forward curve, and calculating the impact of this perturbation on the value of the portfolio.

Risk management under SABR

- Likewise, the vega risk is the sensitivity of the portfolio to volatility and is traditionally measured as the derivative of the option price with respect to the implied volatility.
- The choice of volatility model impacts not only the prices of (out of the money) options but also, at least equally significantly, their risk sensitivities.
- One has to take into account the following issues:
 - (i) How is the vega risk defined: as the sensitivity to the lognormal volatility, normal volatility, or another volatility parameter?
 - (ii) How is the delta risk defined: which volatility parameter should be kept constant when taking the derivative with respect to the underlying?
- Since we assume the SABR model specification, and thus the relevant volatility parameter is the beta vol σ_0 .

Risk management under SABR

- The delta risk of an option is calculated by shifting the current value of the underlying while keeping the current value of implied volatility σ_0 fixed.
- In the case of a caplet / floorlet or a swaption, this amounts to shifting the relevant forward rate without changing the implied volatility:

$$\begin{aligned} F_0 &\rightarrow F_0 + \Delta F_0, \\ \sigma_0 &\rightarrow \sigma_0, \end{aligned} \tag{9}$$

where ΔF_0 is, say, -1 bp.

- Assuming the normal model for valuation, this scenario leads to the option delta:

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial F_0}. \tag{10}$$

Risk management under SABR

- The first term on the right hand side in the formula above is the original Black model delta, and the second arises from the systematic change in the implied (normal) volatility as the underlying changes.
- This formula shows that, in stochastic volatility models, there is an interaction between classic Black-Scholes style greeks!
- The classic delta and vega contribute both to the smile adjusted delta.

Risk management under SABR

- This way of calculating the delta risk is practical for a single option only.
- If our task is to hedge a portfolio of caps / floors and swaptions (of various expirations, strikes and underlyings), we should follow the approach of the beginning of these lecture notes.
- Namely, we subject the portfolio to a number of forward rate shocks and replicate the resulting risk profile with the risk profile of a portfolio of liquid swaps, FRAs, etc.
- This simply means replacing the first of the shifts (9) by the corresponding partial shift of the OIS / forward curve.
- In the following discussion we will implicitly mean these partial shifts while (for the sake of conceptual simplicity) we talk about shifting a single forward rate.

Delta risk under SABR

- The issue with scenario (9) is, however, that it is incompatible with the stochastic volatility dynamics of SABR.
- Namely, decomposing the Brownian motion Z into a component proportional to W and a component W^\perp independent of W , we can write the dynamics in the form:

$$\begin{aligned}dF(t) &= \sigma(t) C(F(t)) dW(t), \\d\sigma(t) &= \alpha\sigma(t) (\rho dW(t) + \sqrt{1 - \rho^2} dW^\perp(t)).\end{aligned}\tag{11}$$

- This shows that

$$d\sigma(t) = \frac{\rho\alpha}{C(F(t))} dF(t) + \text{independent noise}.\tag{12}$$

Delta risk under SABR

- As a result, for non-zero correlation ρ , a move in the forward tends to move the volatility parameter by an amount proportional to the change in the forward.
- This leads us to the conclusion that a scenario consistent with the dynamics is of the form:

$$\begin{aligned} F_0 &\rightarrow F_0 + \Delta F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \delta_F \sigma_0. \end{aligned} \tag{13}$$

Delta risk under SABR

- Here

$$\delta_F \sigma_0 = \frac{\rho \alpha}{F_0^\beta} \Delta F_0 \quad (14)$$

is the expected change in σ_0 caused by the change in the underlying forward.

- The new delta risk [3] is given by

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma_n} \left(\frac{\partial \sigma_n}{\partial F_0} + \frac{\partial \sigma_n}{\partial \sigma_0} \frac{\rho \alpha}{F_0^\beta} \right). \quad (15)$$

- This delta risk, the “Bartlett delta”, respects the dynamics of SABR.

Delta risk under SABR

- Figure 1 shows the classic SABR delta corresponding to three different calibrations: $\beta = 0$ (black line), $\beta = 0.5$ (red line), and $\beta = 1$ (green line):

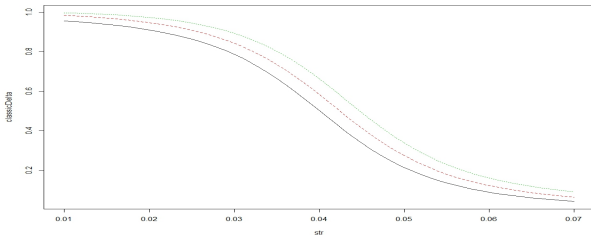


Figure: 1. Classic SABR delta for different values of β .

- Even though all three sets of parameters closely fit the market smile, they lead to different conventional hedges, especially near the money.
- Choosing the incorrect beta can lead to good fits of the smile, but relatively poor delta hedges.

Delta risk under SABR

- Figure 2 shows Bartlett's deltas for the same three sets of parameters:

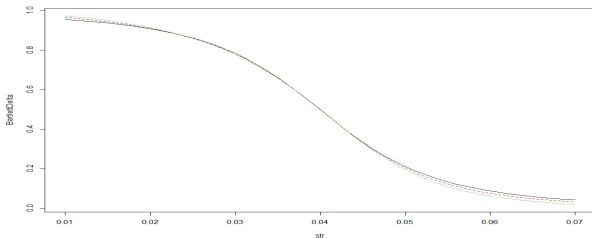


Figure: 2. Bartlett's SABR delta for different values of β .

- Bartlett's delta is nearly independent of β . It depends mainly on the actual market skew / smile, and not on how the smile is parameterized.
- Bartlett's deltas tends to provide more robust hedges.

Delta risk under SABR

- Figure 3 below (taken from [1]) presents empirical data illustrating the presence of the Bartlett correction in the 1Y into 10Y swaption deltas

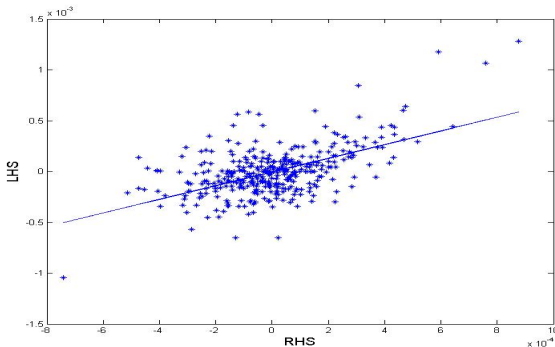


Figure: 3. Regression of $\delta_F \sigma_0$ against $\rho \alpha / F^\beta$ for the 1Y into 10Y swaption $\beta = 0.5$.

Delta risk under SABR

- Figure 4 below (taken from [1]) presents empirical data illustrating the presence of the Bartlett correction in the 5Y into 5Y swaption deltas.

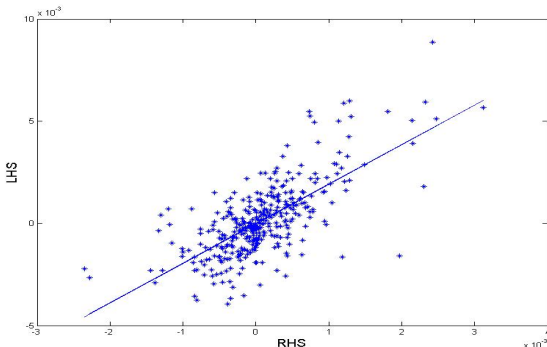


Figure: 4. Regression of $\delta_F \sigma_0$ against $\rho \alpha / F^\beta$ for the 5Y into 5Y swaption $\beta = 0.75$.

Delta risk under SABR

- The robustness of Bartlett's delta can be confirmed by explicit calculations using the asymptotic expression for the implied volatility in the SABR model.
- As a result of these calculations [4], we find that the modified SABR delta is given by

$$\Delta^{\text{mod}} = \frac{\partial V}{\partial F} + \frac{\partial V}{\partial \sigma} \eta + O(F - K), \quad (16)$$

where

$$\eta = \left. \frac{\partial \sigma^{\text{imp}}(T, F, K, \sigma)}{\partial K} \right|_{K=F}. \quad (17)$$

is the skew.

Delta risk under SABR

- Note that, to the leading order in the option moneyness, this expression is independent of the details of the backbone function $C(F)$, it only depends on the implied volatility for the strike K and the skew η .
- Both of these quantities are market observable, and the calibrated model fits them.
- This explains the empirical observation discussed above that the modified SABR delta is practically insensitive to the choice of the parameter β , once the remaining parameters have been optimized.
- In particular, the expression above shows that the modified delta of an at the money option, $K = F$, is independent of the choice of β .

Delta risk under SABR

- This is to be contrasted with the behavior of the classic SABR delta

$$\Delta = \frac{\partial V}{\partial F} + \frac{\partial V}{\partial \sigma} \left(\eta + \frac{\rho \alpha}{C(F)} \right) + O(F - K). \quad (18)$$

- In other words, the classic SABR delta, and thus the corresponding hedging strategy, depends on the choice of the backbone function $C(F)$.

Vega risk under SABR

- Similarly, the vega risk is calculated from

$$\begin{aligned}F_0 &\rightarrow F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \Delta\sigma_0,\end{aligned}\tag{19}$$

and is given by

$$\Lambda = \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma_0}.\tag{20}$$

These formulas are the classic SABR greeks.

- Modified SABR greeks below attempt to make a better use of the model dynamics.
- Since the processes for σ and F are correlated, whenever F changes, on average σ changes as well.
- This change is proportional to the correlation coefficient ρ between the Brownian motions driving F and σ .

Bartlett vega risk

- The vega risk could be calculated from the scenario:

$$\begin{aligned}F_0 &\rightarrow F_0 + \delta_\sigma F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \Delta\sigma_0,\end{aligned}\tag{21}$$

where

$$\delta_\sigma F_0 = \frac{\rho F_0^\beta}{\alpha} \Delta\sigma_0\tag{22}$$

is the average change in F_0 caused by the change in the beta vol.

- This leads to the modified vega risk

$$\Lambda = \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma_0} + \left(\frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial F_0} + \frac{\partial V}{\partial F_0} \right) \frac{\rho F_0^\beta}{\alpha} .\tag{23}$$

Bartlett vega risk

- The first term on the right hand side of the formula above is the classic SABR vega, while the second term accounts for the change in volatility caused by the move in the underlying forward rate.
- Note that part of the option's volatility sensitivity can be viewed as a component of its delta or its vega, depending on risk management approach.
- We take the view that it should be allocated to the delta risk, as monitoring and executing delta hedges are generally easier than vega hedges.

Delta risk under SABR-LMM

- The arguments above can be extended to the SABR-LMM model. Consider a LIBOR forward L_j , and write the corresponding in the form:

$$\begin{aligned}dL_j(t) &= \sigma_j(t) L_j(t)^{\beta_j} (\{\dots\} dt + dW_j(t)), \\d\sigma_j(t) &= \alpha_j(t) \sigma_j(t) (\{\dots\} dt + r_{jj} dW_j + \sqrt{1 - r_{jj}^2} dW_j^\perp(t)),\end{aligned}\tag{24}$$

where $\{\dots\}$ stands for expressions whose explicit form is of no relevance for this calculation.

- This shows that

$$d\sigma_j(t) = \frac{r_{jj}\alpha_j(t)}{L_j(t)^{\beta_j}} dL_j(t) + \text{independent noise} + \{\dots\} dt.\tag{25}$$

- The drift term $\{\dots\} dt$ above is small relative to the first term on the right hand side of the equation above, and we will neglect it.

Delta risk under SABR-LMM

- This shows that it is natural to follow a shift in the LIBOR forward rate:

$$L_{j0} \rightarrow L_{j0} + \Delta L_{j0} \quad (26)$$

by the following shift in the corresponding volatility parameter:

$$\sigma_{j0} \rightarrow \sigma_{j0} + \frac{r_{jj}\alpha_j}{L_{j0}^{\beta_j}} \Delta L_{j0}, \quad (27)$$

where $\alpha_j = \sqrt{\frac{1}{T_m} \int_0^{T_m} \alpha_j(t)^2 dt}$ is the average instantaneous vol of vol.

- This results in the following “Bartlett correction” to the portfolio sensitivity to the shift (26) of the LIBOR forward:

$$\frac{\partial \Pi}{\partial \sigma_{j0}} \frac{r_{jj}\alpha_j}{L_{j0}^{\beta_j}} \Delta L_{j0}. \quad (28)$$

Risk management of mortgages

- We consider a portfolio of CMOs whose current value is equal to Π . The key risks are:
 - (i) Mortgage rate risk
 - (ii) Interest rates risk
 - (iii) Volatility risk
- The mortgage rate risk of the portfolio is its sensitivity to the CMM curve \mathcal{M}_0 :

$$\delta_M \Pi (T) = \frac{\delta \Pi}{\delta \mathcal{M}_0 (T)} .$$

Risk management of mortgages

- In practice, this risk is measured either by
 - (i) shifting the entire CMM curve in parallel by ε and computing the finite difference

$$\delta_M^{\parallel} \Pi = \frac{\Pi(\mathcal{M}_0 + \varepsilon) - \Pi(\mathcal{M}_0 - \varepsilon)}{2\varepsilon},$$

- (ii) or dividing up the curve \mathcal{M}_0 into segments (“buckets”) \mathcal{M}_0^i , and computing the corresponding partial finite difference $\delta_M^i \Pi(t)$.
- (ii) Note that the latter approach amounts to:

$$\delta_M^i \Pi \approx \frac{1}{b-a} \int_a^b \delta_M \Pi(T) dT,$$

where a and b indicate the start and end points of the bucket.

Risk Management of Mortgages

- The risk profile of a TBA is particularly easy to understand within the empirical model:

$$\begin{aligned}\frac{\partial P_C(T)}{\partial M_0(T)} &= D_C(T) P_C(T), \\ \frac{\delta P_C(T)}{\delta f_0(t)} &= D_C(T) P_C(T) \frac{\delta M_0(T)}{\delta f_0(t)}, \\ \frac{\delta P_C(T)}{\delta \sigma_0(t, \tau)} &= D_C(T) P_C(T) \frac{\delta M_0(T)}{\delta \sigma_0(t, \tau)}.\end{aligned}$$

- In other words, within the empirical model:
 - (i) TBA's sensitivity to the mortgage rate is given by its duration.
 - (ii) The sensitivities to the interest rates and their volatility are given by the TBA's duration multiplied by the relevant sensitivity of the mortgage rate.

Risk Management of Mortgages

- Let us compute the hedge ratios of the portfolio to each of the TBAs:

$$\begin{aligned}\frac{\delta \Pi}{\delta P_C(T)} &= \frac{\delta M_0(t)}{\delta P_C(T)} \frac{\delta \Pi}{\delta M_0(t)} \\ &= w_C(T) \delta_M \Pi(t),\end{aligned}$$

where $w_C(T)$ are the weights from the replication formula.

- In practice, we face two issues:
 - Only few coupons are liquidly traded.
 - Only TBAs with settlements out to a few months from today are traded.

Risk management of mortgages

- (i) Let C_1, \dots, C_m denote the available coupons, and calculate the hedge ratios:

$$w_1(T) = \int_0^{(C_1+C_2)/2} w_C(T) dC,$$

$$w_i(T) = \int_{(C_{i-1}+C_i)/2}^{(C_i+C_{i+1})/2} w_C(T) dC, \text{ for } i = 2, \dots, m-1,$$

$$w_m(T) = \int_{(C_{m-1}+C_m)/2}^{\infty} w_C(T) dC.$$

- (ii) The risk to CMM rates with settlement longer than the longest traded PSA is projected onto the short end of the CMM curve.

Risk management of mortgages

- Other risks are computed analogously, as sensitivities to:

- (i) Current forward curve:

$$\delta_f \Pi(T) = \frac{\delta \Pi}{\delta f_0(T)} .$$

- (ii) Current volatilities:

$$\delta_\sigma \Pi(T, \tau) = \frac{\delta \Pi}{\delta \sigma_0(T, \tau)} .$$

- We hedge a portfolio of MBS in the following steps.

- (i) Calculate the portfolio's sensitivity to the CMM rate. Using the dynamic replication methodology, compute the face values of the TBA's required to offset that risk.
- (ii) Calculate the vega risk of the combined portfolio of the MBSs and TBAs, and offset that risk by a position in swaptions and caps / floors.
- (iii) Calculate the delta risk of the combined portfolio of MBSs, TBAs, and interest rate options, and offset that risk by a position in LIBOR swaps, FRAs, and Eurodollar futures.

Broad based risk metrics

- Sensitivity risk measures provide detailed information about the risk of a portfolio and are particularly useful to the management team in charge of the portfolio. Their goal is to provide a myopic view of the risk.
- Additionally, sensitivity based risk metrics measure the risk exposure to typical (i.e. small) market volatility.
- Broad based risk metrics summarize portfolio risk in a way that is easily communicable to the upper management of a financial institution, investors, and regulatory agencies.
- Desired features of such risk metrics are:
 - (i) Coherence
 - (ii) Elicitability

Coherent risk metrics

- Consider a random variable L which may be thought of as a potential loss in a portfolio. A *coherent risk metric* is a numerical value $\rho(L)$ assigned to L in such a way that:
 - (i) *Monotonicity*: if $L_1 \leq L_2$, then $\rho(L_1) \leq \rho(L_2)$.
 - (ii) *Subadditivity*: $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$.
 - (iii) *Positive homogeneity*: $\rho(\alpha L) = \alpha\rho(L)$, for $\alpha > 0$.
 - (iv) *Translation invariance*: If A is deterministic, then $\rho(L + A) = \rho(L)$.
- Note that subadditivity requirement reflects the impact of diversification on portfolio risk: the risk of a combined portfolio does not exceed the sum of risks of parts of the portfolio.

Broad based risk metrics

- In practice, portfolio risk metrics are calculated in one of the following ways:
 - (i) Assuming a parametric form of the probability distribution of L .
 - (ii) Using the historical market data, i.e. using the empirical probability distribution.
 - (iii) Generating Monte Carlo scenarios.
- Reported values of the risk metrics are routinely compared to the actual outcomes: this process is called *back testing*.
- For back testing to be possible, the risk metric should be elicitable, i.e. it should be estimable from the market observations.

VaR and ES

- Two commonly used are:
 - (i) *Value at Risk* (VaR) defined as the critical portfolio loss, such that the probability that the portfolio loss exceeds this value, over the given time horizon, is a given confidence level q . Typically, the industry uses $q = 95%$ or $q = 99%$.
 - (ii) *Expected Shortfall* (ES), a.k.a. *Conditional Value at Risk* (CVaR) is the expected value of portfolio loss provided that it has exceeded VaR_q .
- VaR is known to violate the subadditivity requirement (and is thus not a coherent risk metric), while Expected Shortfall does not appear elicitable.
- The industry is in transition from using VaR in favor of using Expected Shortfall.

VaR and ES

- VaR is a popular and very useful quantitative metric used summarizing for summarizing portfolio risk.
- Let Φ denote the probability distribution of portfolio losses L . Given a significance level α (say, $\alpha = 1\%$) it is defined as

$$\begin{aligned} \text{VaR}_\alpha(L) &= \sup_l \{l : \mathbb{P}(L \geq l) \geq \alpha\} \\ &= \sup_l \{l : \Phi(l) \leq 1 - \alpha\}. \end{aligned}$$

The number $1 - \alpha$ is referred to as the *confidence level*.

VaR and ES

- If the cumulative distribution is continuous and monotone increasing, then $VaR_{\alpha}(L) = \Phi^{-1}(1 - \alpha)$. For the standard normal distribution, the 99% VaR is $N^{-1}(0.99) = 2.326347874$.
- VaR's reliability is often put into question, as it says nothing about what lies above the assumed confidence level. Overreliance on VaR leads to false sense of complacency.
- In particular, the subadditivity is a natural requirement stating that the risk of a combined portfolio cannot exceed the sum of the risks of its parts.
- It reflects the fact that diversification may offset parts of the risks inherent in the portfolio.
- In general, VaR *does not* have the subadditivity property. Exception is the normal distribution (and more generally, elliptical distributions), for which VaR is subadditive.

VaR and ES

- Conditional Value at Risk (CoVaR), a.k.a. Expected Shortfall (ES) is defined as

$$\begin{aligned} \text{CoVaR}_\alpha(L) &= E[L \mid L \geq \text{VaR}_\alpha] \\ &= \frac{1}{\alpha} \int_{\text{VaR}_\alpha}^{\infty} l d\Phi(l) \end{aligned}$$

- If the CDF Φ is continuous, then CoVaR can alternatively be expressed as

$$\text{CoVaR}_\alpha(L) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\gamma(L) d\gamma,$$

VaR and ES

- To see this, we use the fact that $VaR_\gamma(L) = \Phi^{-1}(1 - \gamma)$, and so

$$\begin{aligned}\int_0^\alpha VaR_\gamma(L) d\gamma &= \int_0^\alpha \Phi^{-1}(1 - \gamma) d\gamma \\ &= - \int_\infty^{VaR_\alpha} l \Phi'(l) dl \\ &= \int_{VaR_\alpha}^\infty l d\Phi(l),\end{aligned}$$

where we have made the substitution $l = \Phi^{-1}(1 - \gamma)$, i.e. $\gamma = 1 - \Phi(l)$.

- CoVaR is a coherent risk measure. It is, however, harder to measure than VaR.
- For the standard normal distribution, the 99% CoVaR is 2.66521422.







Stress testing

- Among other broad risk metrics used in the industry are *stress tests*.
- These are required by the regulatory agencies to assure that the financial institutions are adequately capitalized for the event of a large market dislocation (e.g. the recent CCAR stress testing).
- There exists no standard systematic methodology for generation of stress tests.

Generally, two approaches as followed:

- (i) Historical scenarios.
- (ii) Hypothetical scenarios.

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