Interest Rate and Credit Models

10. Term structure models: short rate models

Andrew Lesniewski

Baruch College
New York

Spring 2019

Outline

Term structure modeling

- The real challenge in modeling interest rates is the existence of a term structure of interest rates embodied in the shape of the forward curve.
- Fixed income instruments typically depend on a segment of the forward curve rather than a single point.
- Pricing such instruments requires thus a model describing a stochastic time evolution of the entire forward curve.
- There exists a large number of term structure models based on different choices of state variables parameterizing the curve, number of dynamic factors, volatility smile characteristics, etc.

Term structure modeling

- Time permits us to discuss term structure modeling only in its crudest outline, and we focus on two approaches.
- *Short rate models*, in which the stochastic state variable is taken to be the instantaneous forward rate. Historically, these were the earliest successful term structure models.
- We shall focus on a tractable Gaussian model, namely Vasicek's model and its descendants
- *LIBOR market model*, in which the stochastic state variable is the entire forward curve represented and as a collection of benchmark LIBOR forward rates.
- These, more recently developed, models are descendants of the HJM model and have been popular among practitioners.

Term structure modeling

- Short rates models use the instantaneous spot rate $r(t)$ as the basic state variable.
- In the LIBOR / OIS framework, the short rate is defined as $r(t) = f(t,t)$, where $f(t,s)$ denotes the instantaneous discount (OIS) rate, as explained in Lecture Notes #1.
- The stochastic dynamics of the short rate $r(t)$ is driven by a number of random factors, usually one, two, or three, which are modeled as Brownian motions.
- Depending on the number of these stochastic drivers, we refer to the model as one-, two- or three-factor.
- The stochastic differential equations specifying the dynamics are typically stated under the spot measure.

Term structure modeling

- In the one-factor case the dynamics has the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{1}$$

  where A and B are suitably chosen drift and diffusion coefficients, and W is the Brownian motion driving the process.
- Various choices of the coefficients μ and σ lead to different dynamics of the instantaneous rate. You should consult the literature cited at the end of these notes for a complete catalog of choices available in the repertoire.
- We shall focus on the Vasicek model and its descendant, the Hull-White model.
- In a *multi-factor model* the rate $r(t)$ is represented as the sum of deterministic component and several stochastic components, each of which describes the evolution of a stochastic factor.
- The factors are specified so that the combined dynamics captures closely observed interest rate curve behavior.

Modeling mean reversion of rates

- The simplest term structure model of any practical significance is *Vasicek's model*.
- Under the spot measure Q_0, its dynamics is given by:

$$dr(t) = \lambda(\mu - r(t))\,dt + \sigma dW(t)\,, \tag{2}$$

together with the initial condition:

$$r(0) = r_0. \tag{3}$$

- Originally, this process has been studied in the physics literature, and is known as the *Ornstein - Uhlenbeck process*.

Modeling mean reversion of rates

- A special feature of Vasicek's model is that the stochastic differential equation (2) has a closed form solution.
- In order to find it we utilize the method of variations of constants.
- The homogeneous equation

$$dr\,(t) = -\lambda r\,(t)\,dt$$

has the obvious solution:

$$r\,(t) = Ce^{-\lambda t}, \tag{4}$$

with C an arbitrary constant.

- We seek a particular solution to the inhomogeneous equation in the form of (4) with the constant C replaced by an unknown function $\psi\,(t)$,

$$r_1\,(t) = \psi\,(t)\,e^{-\lambda t}.$$

Modeling mean reversion of rates

- We find readily that $\psi(t)$ has to satisfy the ordinary differential equation:

$$d\psi(t) = \lambda\mu e^{\lambda t} dt + \sigma e^{\lambda t} dW(t)\,.$$

- Consequently,

$$\psi(t) = \mu e^{\lambda t} + \sigma \int_0^t e^{\lambda s} dW(s)\,.$$

- The solution to our problem is the sum of the solution (4) with $C = r_0 - \mu$ (in order to enforce the initial condition) and the particular solution $r_1(t)$:

$$r(t) = r_0 e^{-\lambda t} + \mu\left(1 - e^{-\lambda t}\right) + \sigma \int_0^t e^{-\lambda(t-s)} dW(s)\,. \tag{5}$$

Modeling mean reversion of rates

- To understand better the meaning of this solution, we note that the expected value of the instantaneous rate $r(t)$ is

$$\mathrm{E}^{\mathrm{Q}_0}[r(t)] = r_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}). \tag{6}$$

- Its variance is

$$\mathrm{Var}[r(t)] = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t}). \tag{7}$$

- This means that, on the average, as $t \to \infty$, $X(t)$ tends to μ, and this limit is approached exponentially fast. This property is referred to as *mean reversion* of the short rate.
- The rate of mean reversion is equal to λ, and the time scale τ on which it takes place is given by the inverse of λ, $\tau = 1/\lambda$.

Modeling mean reversion of rates

- Random fluctuations interfering with the mean reversion are of the order of magnitude $\sigma/\sqrt{2\lambda}$.
- This tends to zero, as $\lambda \to \infty$, and thus strongly mean reverting processes are characterized by low volatility.
- Elegant and simple as it is, the Vasicek model has a number of serious shortcomings:
  (i) It is impossible to fit the entire forward curve as the initial condition.
  (ii) There is one volatility parameter only available for calibration (two, if you count the mean reversion rate). That makes fitting the volatility structure virtually impossible.
  (iii) The model is *one-factor*, meaning that there is only one stochastic driver of the process.
  (iv) With non-zero probability, rates may become negative (typically, this probability is fairly low).

One-factor Hull-White model

- Some of these shortcomings can be easily overcome by means of a slight extension of the model.
- A suitable generalization of the Ornstein-Uhlenbeck process (2) is a process which mean reverts to a time dependent level $\mu(t)$ rather than a constant μ.
- Such a process is given by

$$dr(t) = \left(\frac{d\mu(t)}{dt} + \lambda(\mu(t) - r(t))\right) dt + \sigma(t)\, dW(t)\,, \tag{8}$$

where we have also allowed σ to be a function of time.

One-factor Hull-White model

- The presence of the time derivative of $\mu(t)$ in the drift is somewhat surprising.
- However, solving (8) (using again the method of variation of constants) yields

$$r(t) = r_0 e^{-\lambda t} + \mu(t) - \mu(0)\, e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} \sigma(u)\, dW(u)\,, \tag{9}$$

and thus

$$\mathsf{E}^{\mathsf{Q}_0}[r(t)] = r_0 e^{-\lambda t} + \mu(t) - e^{-\lambda t} \mu(0)\,, \tag{10}$$

$$\mathrm{Var}[r(t)] = \int_0^t e^{-2\lambda(t-u)} \sigma(u)^2\, du. \tag{11}$$

One-factor Hull-White model

- That shows that $\mathrm{E}[r(t)] - \mu(t) \to 0$, as $t \to \infty$.
- Note that (9) implies that

$$r(t) = r(s)\, e^{-\lambda(t-s)} + \mu(t) - \mu(s)\, e^{-\lambda(t-s)} + \int_s^t e^{-\lambda(t-u)} \sigma(u)\, dW(u), \quad (12)$$

  for any $s < t$.
- We shall use this fact in the following.

One-factor Hull-White model

- Let us now choose $\mu(t)$ so that $\mu(0) = r_0$. We denote $r_0(t) = \mu(t)$, the current short rate, and write

$$dr(t) = \Big(\frac{dr_0(t)}{dt} + \lambda\big(r_0(t) - r(t)\big)\Big)dt + \sigma(t)\, dW(t)\,. \tag{13}$$

  This process is called the *extended Vasicek* (or *Hull-White*) model.
- We will show later how to choose $r_0(t)$.
- From (12),

$$r(t) = r_0(t) + e^{-\lambda(t-s)}(r(s) - r_0(s)) + \int_s^t e^{-\lambda(t-u)}\sigma(u)\, dW(u)\,, \tag{14}$$

  and so the instantaneous rate is represented as a contribution from the current rate plus a random perturbation.

One-factor Hull-White model

- This representation of $r(t)$ implies that

$$\mathrm{E}_s^{\mathrm{Q}_0}[r(t)] = r_0(t) + e^{-\lambda(t-s)}(r(s) - r_0(s)). \tag{15}$$

In particular, choosing $s = 0$ in (14) we obtain

$$r(t) = r_0(t) + \int_0^t e^{-\lambda(t-u)}\sigma(u)\,dW(u). \tag{16}$$

- The instantaneous 3 month LIBOR rate $r_{\mathrm{3ML}}(t)$ is given by

$$r_{\mathrm{3ML}}(t) = r(t) + b(t), \tag{17}$$

where $b(t)$ is the basis between the instantaneous LIBOR and OIS rates.
- As usual, for simplicity of exposition we assume that the basis curve is given by a deterministic function rather than a stochastic process.

Two-factor Hull-White model

- In the two-factor Hull-White model, the instantaneous rate is represented as the sum of
  (i) the current rate $r_0(t)$, and
  (ii) two stochastic state variables $r_1(t)$ and $r_2(t)$.
- In other words, $r(t) = r_0(t) + r_1(t) + r_2(t)$.
- A natural interpretation of these variables is that $r_1(t)$ controls the levels of the rates, while $r_2(t)$ controls the steepness of the forward curve.

Two-factor Hull-White model

- We assume the stochastic dynamics:

$$\begin{aligned} dr_1(t) &= -\lambda_1 r_1(t)\, dt + \sigma_1(t)\, dW_1(t)\,, \\ dr_2(t) &= -\lambda_2 r_2(t)\, dt + \sigma_2(t)\, dW_2(t)\,, \end{aligned} \tag{18}$$

  where $\sigma_1(t)$ and $\sigma_2(t)$ are the instantaneous volatilities of the state variables $r_1(t)$ and $r_2(t)$, respectively.
- The two Brownian motions are correlated,

$$\mathrm{E}\left[dW_1(t)\, dW_2(t)\right] = \rho\, dt. \tag{19}$$

- The correlation coefficient ρ is typically a large negative number ($\rho \sim -0.9$) reflecting the fact that steepening curve moves tend to correlate negatively with parallel moves.

The zero coupon bond in the Hull-White model

- The key to all pricing is the ability to compute the forward price of a zero coupon bond $P(t, T)$.
- It is given by the expected value of the stochastic discount factor,

$$P(t, T) = \mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T r(u)du}\right], \tag{20}$$

where the subscript t indicates conditioning on $\mathcal{F}_t$.
- Within the Hull-White model (and thus, as a special case, in the Vasicek model), this expected value can be computed in closed form!

The zero coupon bond in the Hull-White model

- Let us consider the one-factor case. We proceed as follows:

$$
\begin{aligned}
&\mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T r(u)du}\right] \\
&\quad = \mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T\left(r_0(u)+e^{-\lambda(u-t)}(r(t)-r_0(t))+\int_t^u e^{-\lambda(u-s)}\sigma(s)dW(s)\right)du}\right] \\
&\quad = e^{-\int_t^T r_0(u)du-h_\lambda(T-t)(r(t)-r_0(t))}\,\mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T\int_t^u e^{-\lambda(u-s)}\sigma(s)dW(s)du}\right],
\end{aligned}
$$

where

$$
h_\lambda(t)=\frac{1-e^{-\lambda t}}{\lambda}\,.
$$

- Integrating by parts we can transform the double integral in the exponent into a single integral

$$
\int_t^T\int_t^u e^{-\lambda(u-s)}\sigma(s)\;dW(s)\;du=\int_t^T h_\lambda(T-s)\,\sigma(s)\,dW(s)\,.
$$

The zero coupon bond in the Hull-White model

- Finally, using the fact that

$$\mathrm{E}_t\left[e^{\int_t^T \varphi(s)dW(s)}\right] = e^{\frac{1}{2}\int_t^T \varphi(s)^2 ds}, \tag{21}$$

we obtain the following expression for the price of a zero coupon bond:

$$P(t,T) = A(t,T)\, e^{-h_\lambda(T-t)r(t)}. \tag{22}$$

- Here

$$A(t,T) = e^{-\int_t^T r_0(u)du + h_\lambda(T-t)r_0(t) + \frac{1}{2}\int_t^T h_\lambda(T-s)^2\sigma(s)^2 ds}. \tag{23}$$

- Note in particular that the discount factor $P_0(0,T)$ has the form

$$\begin{aligned} P_0(0,T) &= P(0,T) \\ &= e^{-\int_0^T r_0(s)ds + \frac{1}{2}\int_0^T h_\lambda(T-s)^2\sigma(s)^2 ds}. \end{aligned} \tag{24}$$

The zero coupon bond in the Hull-White model

- The generalization of formula of (22) to the case of the two-factor Hull-White model reads:

$$P(t,T)=A(t,T)\,e^{-h_{\lambda_1}(T-t)r_1(t)-h_{\lambda_2}(T-t)r_2(t)}. \tag{25}$$

- Here,

$$\begin{aligned}A(t,T)&=e^{-\int_t^T r_0(u)du}\\&\times e^{\frac{1}{2}\int_t^T\left(h_{\lambda_1}(T-s)^2\sigma_1(s)^2+2\rho h_{\lambda_1}(T-s)h_{\lambda_2}(T-s)\sigma_1(s)\sigma_2(s)+h_{\lambda_2}(T-s)^2\sigma_2(s)^2\right)ds}.\end{aligned} \tag{26}$$

Options on a zero coupon bond

- Using the above expressions for the zero coupon bond, it is possible to derive explicit, closed form expressions for the valuation of European options on such bonds.
- The calculations are elementary, if a bit tedious, and we shall defer them to the next homework assignment.
- We focus on the one factor Hull-White model; it is straightforward to extend these calculations to the two factor model.
- Consider a call option struck at K and expiring at T on a zero coupon bond maturing at $T_{\text{mat}} > T$. Then, its price is equal to

$$\mathrm{PV}_{\text{call}} = P_0(0, T_{\text{mat}})\, N(d_+) - KP_0(0, T)\, N(d_-)\,, \tag{27}$$

where

$$d_{\pm} = \frac{1}{\overline{\sigma}} \log \frac{P_0(0, T_{\text{mat}})}{P_0(0, T)\, K} \pm \frac{\overline{\sigma}}{2}\,, \tag{28}$$

Options on a zero coupon bond

- Here, as usual,

$$\overline{\sigma} = \left(\int_0^T e^{-2\lambda(T-s)} \sigma(s)^2 \, ds \right)^{1/2} h_\lambda (T_{\mathrm{mat}} - T). \tag{29}$$

- Similarly, the price of a put struck at K is given by

$$\mathrm{PV}_{\mathrm{put}} = K P_0(0, T) N(-d_-) - P_0(0, T_{\mathrm{mat}}) N(-d_+). \tag{30}$$

- Since floorlets and caplets can be thought of as calls and puts on FRAs, these formulas can be used as building blocks for valuation of caps and floors in the Hull-White model.
- On the other hand, from the Hull-White model point of view, swaptions are "exotic" instruments, and no closed form valuation formulas are available.

Calibration of the Hull-White model

- A term structure model has to be *calibrated* to the market before it can be used for valuation purposes.
- All the free parameters of the model have to be assigned values, so that the model reprices exactly (or close enough) the prices of a selected set of liquid vanilla instruments.
- In the case of the Hull-White model, this amounts to:
  - (i) Matching the current forward curve, which is accomplished by choosing $r_0(t)$ to match the current instantaneous OIS curve.
  - (ii) Matching the volatilities of selected options.
- These two tasks have to be performed simultaneously.

Calibration of the Hull-White model

- Specifically, in the one-factor model, today's value of the discount factor is

$$P(0,T)=e^{-\int_0^T r_0(s)ds+\frac{1}{2}\int_0^T h_\lambda(T-s)^2\sigma(s)^2 ds}. \qquad (31)$$

- This implies that the instantaneous forward rate is

$$\begin{aligned} f(0,T)&=-\frac{\partial \log P(0,T)}{\partial T}\\ &=r_0(T)-\int_0^T e^{-\lambda(T-s)}h_\lambda(T-s)\sigma(s)^2\,ds, \end{aligned} \qquad (32)$$

and so

$$r_0(t)=f(0,t)+\int_0^t e^{-\lambda(t-s)}h_\lambda(t-s)\sigma(s)^2\,ds. \qquad (33)$$

- In other words, the current short rate $r_0(t)$ is equal to the current instantaneous forward rate plus a (volatility dependent) convexity correction.

Calibration of the Hull-White model

- As a result, the curve data (i.e. $r_0(t)$) are entangled with the dynamic model data (i.e. λ and $\sigma(t)$), and they require joined calibration.
- This phenomenon is typical of all short rate models. We will see later that in the LMM framework, the curve data and vol data can be handled separately.
- It is impossible to calibrate the Hull-White model in such a way that the prices of all caps / floors and swaptions for all expirations, strikes and underlying tenors are matched.
- This is a consequence of:
  (i) the volatility dynamics of the Hull-White model (normal) and the intrinsic smile,
  (i) the paucity of model parameters available for calibration.
- Commonly used calibration strategies are:
  (i) *global optimization*, suitable for a portfolio,
  (ii) deal specific *local calibration*, suitable for an individual instrument.

Calibration of the Hull-White model

- The instantaneous volatility function (or functions, in the multi-factor case) $\sigma(t)$ is assumed to be locally constant.
- That means that we divide up the time axis into a number of subperiods $[T_j,\ T_{j+1})$ and set $\sigma(t) = \sigma_j$, for $t \in [T_j,\ T_{j+1})$.
- Typically, the subdivision is fine in the short end (say, 3 month or shorter periods) and coarser as we move towards the far end.
- Global optimization consists in selecting the parameters σ_j so as to minimize the objective function

$$\mathcal{L}(\sigma) = \frac{1}{2} \sum_{\text{all instruments}} \left(\sigma_{\mathrm{n}}(\sigma) - \overline{\sigma}_{\mathrm{n}}\right)^2, \tag{34}$$

  where $\overline{\sigma}_{\mathrm{n}}$ and $\sigma_{\mathrm{n}}(\sigma)$ are the market and model prices of all calibration instruments, respectively.
- The set of calibration instruments above may include both swaptions and caps / floors of varying expirations, strikes and tenors.

Calibration of the Hull-White model

- Local calibration consists in selecting a set of instruments (swaptions or caps / floors) whose risk characteristics match the risk characteristics of a particular trade.
- For example, in order to model a Bermudan swaption (to be discussed later in the course), one often selects *co-terminal swaptions* of the same strike (not necessarily at the money) as calibrating instruments.
- Co-terminal swaptions are defined as swaptions whose underlying swaps have the same final maturities, e.g. 1Y$\to$ 10Y, 2Y$\to 9Y, \ldots, 10Y \to 1Y$.
- Calibration to co-terminal swaptions is exact, i.e. the model re-prices the calibrating instruments exactly.

Pricing under the Hull-White model

- Despite the simple structure of the Hull-White model, most instruments cannot be priced by means of closed form expressions such as those for caps and floors of the previous section.
- One has to resort numerical techniques. Among them, two are particularly important:
  - (i) Tree methods.
  - (ii) Monte Carlo methods.
- For the details I defer you to literature cited at the end of these notes.

Eurodollar / FRA convexity correction

- As a simple application of the Hull-White model, we shall now derive a closed form expression for the Eurodollar / FRA convexity correction discussed in Lecture Notes #6.
- We know from Lecture Notes #1 that the (currently observed) LIBOR forward rate $L_0\,(T_1, T_2)$ is the expected value of

$$\frac{1}{\delta}\left(\frac{1}{P(t, T_1, T_2)} - 1\right) + B_0(T_1, T_2) \tag{35}$$

  under the T_2-forward measure Q_{T_2}.
- Here $B_0(T_1, T_2)$ denotes the credit spread between LIBOR and OIS.

Eurodollar / FRA convexity correction

- This is, indeed, almost the definition of the T_2-forward measure!
- Consequently, $L_0\left(T_1, T_2\right)$ is given by:

$$
\begin{aligned}
L_0\left(T_1, T_2\right) &= \frac{1}{\delta}\left(\frac{1}{P_0\left(T_1, T_2\right)} - 1\right) + B_0(T_1, T_2) \\
&= \frac{1}{\delta}\left(\frac{P_0\left(0, T_1\right)}{P_0\left(0, T_2\right)} - 1\right) + B_0(T_1, T_2).
\end{aligned}
\tag{36}
$$

- It is easy to calculate this rate within the Hull-White model. Let us first consider the one-factor case.

Eurodollar / FRA convexity correction

- Using (31), we find that

$$
\begin{aligned}
&L_0\left(T_1, T_2\right) \\
&\quad = \frac{1}{\delta}\left(e^{\int_{T_1}^{T_2} r_0(s)ds-\frac{1}{2}\left(\int_0^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds-\int_0^{T_1} h_\lambda(T_1-s)^2\sigma(s)^2ds\right)}-1\right) \qquad (37) \\
&\qquad + B_0(T_1, T_2).
\end{aligned}
$$

- On the other hand, the rate $L_0^{\text{fut}}\left(T_1, T_2\right)$ implied from the Eurodollar futures contract is given by the expected value of (35) under the spot measure Q_0, namely

$$
L_0^{\text{fut}}(T_1, T_2) = \frac{1}{\delta}\left(\mathsf{E}^{\mathsf{Q}_0}\left[e^{\int_{T_1}^{T_2} r(t)dt}\right]-1\right)+B_0(T_1, T_2).
$$

- We have explained this fact in Lecture #4, attributing it to the practice of daily[1] margin account adjustments by the Exchange.

[1] which we model as continuous

Eurodollar / FRA convexity correction

- In order to calculate this expected value we proceed as in the calculation leading to the explicit formula for $P(t, T)$:

$$
\begin{aligned}
&\mathsf{E}^{\mathsf{Q}_0}\left[e^{\int_{T_1}^{T_2} r(t)dt}\right] \\
&= \mathsf{E}^{\mathsf{Q}_0}\left[e^{\int_{T_1}^{T_2}(r_0(t)+\int_0^t e^{-\lambda(t-s)}\sigma(s)dW(s))dt}\right] \\
&= e^{\int_{T_1}^{T_2} r_0(t)dt}\,\mathsf{E}^{\mathsf{Q}_0}\left[e^{\int_0^{T_2} h_\lambda(T_2-s)\sigma(s)dW(s)-\int_0^{T_1} h_\lambda(T_1-s)\sigma(s)dW(s)}\right] \\
&= e^{\int_{T_1}^{T_2} r_0(t)dt}\,\mathsf{E}^{\mathsf{Q}_0}\left[e^{\int_0^{T_1}(h_\lambda(T_2-s)-h_\lambda(T_1-s))\sigma(s)dW(s)+\int_{T_1}^{T_2} h_\lambda(T_2-s)\sigma(s)dW(s)}\right] \\
&= e^{\int_{T_1}^{T_2} r_0(t)dt+\frac{1}{2}\left(\int_0^{T_1}(h_\lambda(T_2-s)-h_\lambda(T_1-s))^2\sigma(s)^2ds+\int_{T_1}^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds\right)}.
\end{aligned}
$$

Eurodollar / FRA convexity correction

- Therefore,

$$
\begin{aligned}
L_0^{\text{fut}}(T_1, T_2) \\
&= \frac{1}{\delta}\left(e^{\int_{T_1}^{T_2} r_0(t)dt + \frac{1}{2}\left(\int_0^{T_1}(h_\lambda(T_2-s)-h_\lambda(T_1-s))^2\sigma(s)^2 ds + \int_{T_1}^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2 ds\right)} - 1\right) \\
&= L_0(T_1, T_2) + \frac{1}{\delta}(1+\delta F_0(T_1, T_2)) \\
&\qquad \times \left(e^{\int_0^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2 ds - \int_0^{T_1} h_\lambda(T_2-s)h_\lambda(T_1-s)\sigma(s)^2 ds} - 1\right),
\end{aligned}
$$

where $F_0(T_1, T_2) = L_0(T_1, T_2) - B_0(T_1, T_2)$ is the forward rate calculated off the OIS curve.

Eurodollar / FRA convexity correction

- Consequently, the Eurodollar / FRA convexity adjustment is given by

$$\begin{aligned}\Delta_{\mathrm{ED\,/\,FRA}}(T_1,T_2) &= \frac{1}{\delta}\left(1+\delta F_0(T_1,T_2)\right)\\ &\quad\times\left(e^{\int_0^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2 ds-\int_0^{T_1} h_\lambda(T_2-s)h_\lambda(T_1-s)\sigma(s)^2 ds}-1\right).\end{aligned} \tag{38}$$

- This expression can be approximated by a much simpler expression, if we expand the exponential function to the first order and neglect all higher order terms.
- We also neglect the terms proportional to $F_0(T_1,T_2))$, as well as the integral $\int_{T_1}^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2\,ds$.
- A moment of reflection shows that all these terms do not, indeed, contribute significantly to $\Delta_{\mathrm{ED\,/\,FRA}}(T_1,T_2)$.

Eurodollar / FRA convexity correction

- As a result we find the following expression for the convexity adjustment:

$$\Delta_{\text{ED / FRA}}(T_1, T_2) \simeq \frac{1}{\delta}\int_0^{T_1} h_\lambda(T_2 - s)\big(h_\lambda(T_2 - s) - h_\lambda(T_1 - s)\big)\sigma(s)^2\, ds. \quad (39)$$

- In the case of a constant instantaneous volatility, $\sigma(t) = \sigma$, the last integral can be evaluated in closed form, and the result is:

$$\begin{aligned}\Delta_{\text{ED / FRA}} \simeq \frac{\sigma^2}{2\lambda^3\delta}\Big(&\big(1 - e^{-2\lambda T_1}\big)\big(1 - e^{-\lambda(T_2 - T_1)}\big)^2 \\ &+ \big(1 - e^{-\lambda(T_2 - T_1)}\big)\big(1 - e^{-\lambda T_1}\big)^2\Big). \end{aligned} \quad (40)$$

- This formula is very easy to implement in computer code.

Eurodollar / FRA convexity correction

- The calculations in the case of the two-factor Hull-White model are similar, if a bit more tedious.
- The corresponding formula reads:

$$\begin{aligned}\Delta_{\mathrm{ED\,/\,FRA}}(T_1, T_2) \simeq \frac{1}{\delta} \sum_{1 \le j,k \le 2} \rho_{jk} \int_0^{T_1} h_{\lambda_j}(T_2 - s) \\ \times \big(h_{\lambda_k}(T_2 - s) - h_{\lambda_k}(T_1 - s)\big)\sigma_j(s)\sigma_k(s)\, ds,\end{aligned} \tag{41}$$

  where $\rho_{11} = \rho_{22} = 1$, $\rho_{12} = \rho_{21} = \rho$.
- Note that for any real value λ, $h_\lambda(s)$ is non-negative and monotone increasing.
- Therefore, the convexity adjustments implied by the Hull-White model are always positive (as they should be!).

Affine term structure models

- *Affine* term structure models are short rate models with the property that the zero coupon bond price is of the form:

$$P(t,T) = A(t,T)\exp\Big(-B(t,T)r(t)\Big). \tag{42}$$

- It follows from (22) that the Hull-White model is an affine model[2].
- In general, one can show that a short rate model (48) is affine if μ and σ^2 are themselves affine functions:

$$\begin{aligned}\mu(t,x) &= a(t)\,x + b(t)\,,\\ \sigma(t,x)^2 &= c(t)\,x + d(t)\,,\end{aligned} \tag{43}$$

with deterministic $a(t)\,,b(t)\,,c(t)\,,d(t)$.

[2] The same is, of course, true for the two factor Hull-White model.

Affine term structure models

- For affine term structure models with affine coefficients μ and σ^2, the functions A and B satisfy the following ordinary differential equations:

$$\begin{aligned}\frac{dB(t,T)}{dt} + a(t)\,B(t,T) - \frac{1}{2}\,c(t)\,B(t,T)^2 + 1 &= 0,\\ \frac{d\log A(t,T)}{dt} - b(t)\,B(t,T) + \frac{1}{2}\,d(t)\,B(t,T)^2 &= 0,\end{aligned} \tag{44}$$

  with

$$\begin{aligned}B(T,T) &= 0,\\ A(T,T) &= 1.\end{aligned} \tag{45}$$

- The first of equations (44) is called *Riccati's equation*, and cannot, in general, be solved in closed form.

The Cox-Ingersoll-Ross model

- The *Cox-Ingersoll-Ross model* (CIR) model is defined by

$$\begin{aligned} dr(t) &= \lambda(\mu - r(t))dt + \sigma\sqrt{r(t)}\,dW(t), \\ r(0) &= r_0, \end{aligned} \tag{46}$$

  under the risk neutral measure.
- Usually the condition $2\lambda\mu > \sigma^2$ is imposed which guarantees that the probability of $r(t)$ reaching the origin is zero.
- The CIR model is affine with

$$\begin{aligned} A(t,T) &= \Big(\frac{2h\exp(\lambda+h)(T-t)/2}{2h+(\lambda+h)\big(\exp(T-t)h-1\big)}\Big)^{2\lambda\mu/\sigma^2}, \\ B(t,T) &= \frac{2\big(\exp(T-t)h-1\big)}{2h+(\lambda+h)\big(\exp(T-t)h-1\big)}, \end{aligned} \tag{47}$$

  where $h = \sqrt{\lambda^2 + 2\sigma^2}$.

The Cox-Ingersoll-Ross model

- As in the case of the Vasicek model, the CIR model cannot be fit to the initial term structure of rates, and is not used in the industry.
- However, the following extended CIR model:

$$\begin{aligned} r(t) &= \theta(t) + x(t), \\ dx(t) &= \lambda(\mu - x(t))dt + \sigma\sqrt{x(t)}\,dW(t), \\ x(0) &= x_0, \end{aligned} \tag{48}$$

  where $\theta(t)$ is deterministic, can be fitted to the initial term structure of rates.
- Also, a two factor version of the extended CIR model is used in the industry.

Short rates models with stochastic volatility

- Other extensions involve short rate term structure models with stochastic volatility.
- For example, the following specification extends the one factor Hull-White model to allow for stochastic volatility (Hull-White SABR model):

$$\begin{aligned} r(t) &= r_0(t) + x(t), \\ dx(t) &= -\lambda x(t)\, dt + \sigma(t)\, v(t)^{\beta}\, dW(t), \\ dv(t) &= \alpha(t) v(t) dZ(t), \end{aligned} \tag{49}$$

where $\sigma(t)$ and $\alpha(t)$ are deterministic, and

$$\begin{aligned} x(0) &= 0, \\ v(0) &= 1. \end{aligned} \tag{50}$$

- The Brownian motions are correlated,

$$dW(t)\, dZ(t) = \chi dt. \tag{51}$$

References

Andersen, L., and Piterbarg, V.: *Interest Rate Modeling*, Vol. 2, Atlantic Financial Press (2010).

Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).

Hull, J.: *Options, Futures and Other Derivatives* Prentice Hall (2005).