Elements of Financial Engineering Course

Baruch-NSD Summer Camp 2019

Lecture 4: Modern Portfolio Theory and CAPM

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Outline of the lecture

- Modern portfolio theory and the Markowitz model
- Capital asset pricing model (CAPM)
- Arbitrage pricing theory (APT)
- Fama-French 3-factor model
- Black-Litterman model
- Merton's problem

Harry Markowitz

From the Wikipedia page in Wikipedia:

- Markowitz won the Nobel Memorial Prize in Economic Sciences in 1990 while a professor of finance at Baruch College of the City University of New York.
- In the preceding year, he received the John von Neumann Theory Prize from the Operations Research Society of America (now Institute for Operations Research and the Management Sciences, INFORMS) for his contributions in the theory of three fields: portfolio theory; sparse matrix methods; and simulation language programming (SIMSCRIPT).

Extensions

From Wikipedia:

Since MPT's introduction in 1952, many attempts have been made to improve the model, especially by using more realistic assumptions.

Post-modern portfolio theory extends MPT by adopting non-normally distributed, asymmetric, and fat-tailed measures of risk. This helps with some of these problems, but not others.

Black-Litterman model optimization is an extension of unconstrained Markowitz optimization that incorporates relative and absolute 'views' on inputs of risk and returns from financial experts. With the advances in Artificial Intelligence, other information such as market sentiment and financial knowledge can be incorporated automatically to the 'views'.

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**Portfolio optimization problem**

The problem of constructing an optimal portfolio or asset allocation, subject to a certain criterion, is usually done in two spaces:

- the holding-price space: the investor adjusts directly the holdings of the assets and observes the P&L of a portfolio.
- the weight-return space: the investor adjusts the weights of his wealth that are invested in each asset, observing the (linear) return of his portfolio.

In either case, the investor is facing a constrained optimization problem.

**Note**

In the Markowitz model, we shall be working in the weight-linear return space.

**Settings in the Markowitz model**

In a single period setting, consider an economy consisting of \( N \) risky assets and a risk free asset. Let \( L = [L_1, \ldots, L_N]' \) be the (random) vector of (annualized) linear returns of risky assets at investment horizon and \( r_f \) the risk free rate, assumed constant.

Denote by \( \ell' \) and \( \Sigma \) the vector of expectation and covariance matrix of the returns of the risky assets, respectively.

Let \( w = [w_1, \ldots, w_N]' \) be the vector of weights of an investor’s wealth that are invested in the risky assets. Positive weight corresponds to long position, whereas negative weight to short position. If the wealth is fully invested in risky assets, then weights sum up to 1, i.e., \( u'w = 1 \), where \( u \) is the vector of ones: \( u = [1, 1, \ldots, 1]' \).

Thus the investor’s portfolio is represented by a vector of weights \( w \).

**Note**

- \( \Sigma \) is assumed nonsingular, therefore \( \Sigma^{-1} \) exists.

**Method of Lagrange multiplier**

Consider the constrained minimization problem

\[
\min_x f(x)
\]

subject to the equality constraint \( g(x) = 0 \). The method of Lagrange multiplier provide an algorithm for solving such problem.

First define the Lagrangian \( L \) as

\[
L(x, \lambda) = f(x) - \lambda g(x).
\]

The critical points is characterized by the first order criterion:

\[
\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0.
\]

Thus, by solving the system of equations that a minimum point must satisfies

\[
D_x f = \lambda D_x g, \\
g(x) = 0.
\]
Minimum variance portfolio (MVP)

The weights $w_{\text{mvp}}$ for minimum variance portfolio is determined by the solution to the constrained optimization problem

$$\min_w w' \Sigma w$$

subject to

$$w' u = 1.$$ 

By applying the method of Lagrange multiplier, we obtain the solution as

$$w_{\text{mvp}} = \frac{\Sigma^{-1} u}{u' \Sigma^{-1} u}.$$ 

The expected return $\ell_{\text{mvp}}$ and the variance $\sigma_{\text{mvp}}^2$ of the minimum variance portfolio are given by

$$\ell_{\text{mvp}} = w_{\text{mvp}}' \ell = \frac{u' \Sigma^{-1} \ell}{u' \Sigma^{-1} u},$$

$$\sigma_{\text{mvp}}^2 = w_{\text{mvp}}' \Sigma w_{\text{mvp}} = \frac{1}{u' \Sigma^{-1} u}.$$ 

Note

- $u' \Sigma^{-1} u$ is equal to the sum of all the entries in $\Sigma^{-1}$. 

Efficient frontier


"A Markowitz-efficient portfolio is one where no added diversification can lower the portfolio’s risk for a given return expectation (alternately, no additional expected return can be gained without increasing the risk of the portfolio). The Markowitz Efficient Frontier is the set of all portfolios that will give the highest expected return for each given level of risk. These concepts of efficiency were essential to the development of the capital asset pricing model (CAPM)."
**Efficient frontier**

For a given expected return $\mu$, determine the weights of the portfolio that has the smallest variance. This is equivalent to solving the constrained optimization problem

$$\min_w w' \Sigma w$$

subject to

$$u'w = 1, \quad w' \ell = \mu.$$  

By applying the method of Lagrange multiplier, the solution to the optimization problem is given by

$$w_{\text{eff}}(\mu) = \frac{B \Sigma^{-1} u - A \Sigma^{-1} \ell}{D} + \frac{C \Sigma^{-1} \ell - A \Sigma^{-1} u}{D} \mu$$

where

$$A = u' \Sigma^{-1} \ell, \quad B = \ell' \Sigma^{-1} \ell, \quad C = u' \Sigma^{-1} u, \quad D = BC - A^2.$$  

**Note**

- $w_{\text{eff}}$ is linear in $\mu$.
- The minimal value is given by $\sigma_{\text{eff}}^2(\mu) = \frac{B - 2A \mu + C \mu^2}{D}$, which is quadratic in $\mu$.

**Definition**

**Efficient frontier** is the graph of the function $\sigma = \sigma_{\text{eff}}(\mu)$, for $\mu \geq \ell_{\text{MVP}}$, in the $\sigma - \mu$ plane.

**Note**

- Any rational investor should hold his portfolio on the efficient frontier.
- At this point, given any two portfolios on the efficient frontier, we are not able to tell if one is superior to the other.

**Efficient frontier with short sell constraint**

For a given expected return $\mu$, determine the weights of the portfolio that has the smallest variance without short selling. This is equivalent to solving the constrained optimization problem

$$\min_w w' \Sigma w$$

subject to

$$u'w = 1, \quad w' \ell = \mu, \quad w \geq 0.$$  

However, the problem unfortunately has no analytical solution but can be efficiently solved numerically by quadratic programming schemes.
**Market portfolio**

We further assume that $r_f < \ell_{MVP}$. In other words, the risk free rate is smaller than the expected return of the MVP.

Now consider the case that an investor would like to reserve part of his wealth in the risk free asset and invest the rest in the risky assets. How should he determine the weights?

One can show that there exists a unique point, name it $M$, with coordinates $(\sigma_M, \ell_M)$ on the efficient frontier such that the line connecting $(0, r_f)$ (noted that risk free asset has zero variance) and $(\sigma_M, \ell_M)$ is tangent to the efficient frontier. Portfolio weights corresponding to the point $M$ is called the *market portfolio* and the tangent line is called the *capital market line* (CML).

**Note**

One can show that the weights $w_M$ of the market portfolio is given by

$$w_M = \frac{\Sigma^{-1}(\ell - r_f u)}{u'\Sigma^{-1}(\ell - r_f u)}.$$

Thus, the expected return $\ell_M$ and the standard deviation $\sigma_M$ are given by

$$\ell_M = \ell'w_M = \frac{\ell'\Sigma^{-1}(\ell - r_f u)}{u'\Sigma^{-1}(\ell - r_f u)},$$
$$\sigma_M = \sqrt{w_M'\Sigma w_M} = \frac{\sqrt{(\ell - r_f u)'\Sigma^{-1}(\ell - r_f u)}}{|u'\Sigma^{-1}(\ell - r_f u)|}.$$

**Capital market line (CML)**

- Those portfolios with expected return and standard deviation lying on the tangent line to the efficient frontier are called *tangent portfolios*.
- For any given tangent portfolio, the following relationship between its expected return $\mu$ and standard deviation $\sigma$ holds. This relationship is called the *capital market line* (CML).

$$\mu = r_f + \frac{\ell_M - r_f}{\sigma_M} \sigma.$$

- Equivalently, we can rewrite the CML in term of Sharpe ratio as follows, which implies that the Sharpe ratios of tangent portfolios are indifferent and are all equal to the Sharpe ratio of the market portfolio.

$$\frac{\mu - r_f}{\sigma} = \frac{\ell_M - r_f}{\sigma_M}.$$

**Sharpe ratio**

Let $\mu$ be the expected return and $\sigma$ be the standard deviation of a portfolio (or a trading strategy). The Sharpe ratio of the portfolio (or the trading strategy) is defined by the ratio

$$\frac{\mu - r_f}{\sigma}.$$

**Note**

Sharpe ratio is a commonly quoted benchmark index for measuring the performance of a portfolio or a trading strategy.
Capital Asset Pricing Model

- The relationship described in CML holds only for tangent portfolios.
- A natural question is: Does there exist similar relationship between the expected return and standard deviation for nontangent portfolios?
- The answer turns out to be yes and the relationship is the celebrated capital asset pricing model (CAPM).

History of CAPM


The CAPM was introduced by Jack Treynor (1961, 1962), William F. Sharpe (1964), John Lintner (1965a,b) and Jan Mossin (1966) independently, building on the earlier work of Harry Markowitz on diversification and modern portfolio theory. Sharpe, Markowitz and Merton Miller jointly received the 1990 Nobel Memorial Prize in Economics for this contribution to the field of financial economics. Fischer Black (1972) developed another version of CAPM, called Black CAPM or zero-beta CAPM, that does not assume the existence of a riskless asset. This version was more robust against empirical testing and was influential in the widespread adoption of the CAPM.

The $\beta$ of a portfolio

Let $r$ be the (random) return of a portfolio and $r_M$ the return of the market portfolio at investment horizon. The portfolio's $\beta$ is defined by the ratio

$$\beta := \frac{\text{cov}(r, r_M)}{\sigma_M^2} = \rho \frac{\sigma}{\sigma_M},$$

where $\rho$ is the correlation coefficient between $r$ and $r_M$.

In particular, if $r_n$ is the linear return of the risky asset $n$, then $\beta_n := \frac{\text{cov}(r_n, r_M)}{\sigma_M^2}$ is the beta of risky asset $n$.

Note

- For a tangent portfolio, i.e., the portfolio lying on the CML, since portfolio weights are given by linear combinations of the weight on risk free asset and the weights of market portfolio, $\rho = 1$. We have $\beta = \frac{\sigma}{\sigma_M}$, which recovers the equation of capital market line.
- This is the $\beta$ that was used in the definition of the Treynor ratio. Recall that the Treynor ratio, in our current notations, is defined as $\frac{\mu - r_f}{\beta}$. 

**CAPM formula**

For a given portfolio with expected return $\mu$ and standard deviation $\sigma$, the following relationships holds

$$\mu - r_f = \beta (\ell_M - r_f),$$

where $\beta = \frac{\text{cov}(r, r_M)}{\text{var}(r)} = \frac{\rho \cdot \sigma}{\sigma_M}$. This relationship is called the CAPM.

Equivalently, we can rewrite the CAPM in terms of Sharpe ratios as

$$\frac{\mu - r_f}{\sigma} = \frac{\ell_M - r_f}{\sigma_M}$$

- In words, the expected excess return of a portfolio is proportional to the expected excess return of the market portfolio.
- In practice, $\beta$ can be estimated by regressing $r$ against the return of market portfolio.
- Since $|\rho| \leq 1$, market portfolio, and in fact all the tangent portfolios, have the highest Sharpe ratio among all portfolios. In other words, in terms of Sharpe ratio it is not possible to beat the market.

**Security market line (SML)**

The relationship between the expected excess return of a portfolio and the expected excess of the market portfolio is called the security market line.

$$\mu - r_f = \beta (\ell_M - r_f).$$

In particular, for $n = 1, \ldots, N$ we have

$$\ell_n - r_f = \beta_n (\ell_M - r_f).$$

**Get our hands dirty - an illustrative example**

In [2]:

```
# load required package quantmod
library(quantmod)
```

In [4]:

```
# download price series from Yahoo Finance
getSymbols(c("AAPL", "GOOG", "AXP", "GM", "PFE", "XOM"), from='2017-01-02')
# getSymbols(c("AAPL", "GOOG", "AXP", "GM", "PFE", "XOM"), from='2015-01-01', to='2018-12-31')
```

Error in curl::curl_download(cu, tmp, handle = h): Timeout was reached: Resolving timed out after 10000 milliseconds

Traceback:

1. getSymbols(c("AAPL", "GOOG", "AXP", "GM", "PFE", "XOM"), from = "2017-01-02")
2. do.call(paste("getSymbols.", symbol.source, sep = ""), list(Symbols = current
 symbols,
   env = env, verbose = verbose, warnings = warnings, auto.assign = auto.assign,
   ..., .haslsym. = .haslsym.))
3. getSymbols.yahoo(Symbols = c("AAPL", "GOOG", "AXP", "GM", "PFE", "XOM"),
   "XOM"), env = <environment>, verbose = FALSE, warnings = TRUE,
   auto.assign = TRUE, from = "2017-01-02", .haslsym. = FALSE)
4. .getHandle(curl.options)
5. new.session()
6. curl::curl_download(cu, tmp, handle = h)
In [4]:

```r
aapl <- AAPL$AAPL.Adjusted
goog <- GOOG$GOOG.Adjusted
axp <- AXP$AXP.Adjusted
xom <- XOM$XOM.Adjusted
gm <- GM$GM.Adjusted
pfe <- PFE$PFE.Adjusted
```
In [5]:
market <- data.frame(aapl, goog, axp, xom, gm, pfe)
head(market, 20)
tail(market)

# summary statistics
summary(market)
## Table 1: Stock Prices

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<tr>
<th>Date</th>
<th>AAPL.Adjusted</th>
<th>GOOG.Adjusted</th>
<th>AXP.Adjusted</th>
<th>XOM.Adjusted</th>
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## Table 2: Summary Statistics

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<td>48.48</td>
<td>59.51</td>
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<td>25.37</td>
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Out[5]:

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<td>Max.</td>
<td>229.39</td>
<td>1287.6</td>
<td>84.56</td>
<td></td>
</tr>
</tbody>
</table>
In [6]:
# number of data points
n <- dim(market)[1]

# number of days in a year
N <- 252

# Daily linear return series
market_linret <- (market[-1]/market[-n] - 1)
head(market_linret)

# summary statistic of annualized linear returns
summary(market_linret)

# calculate sample mean and standard deviations of daily linear returns then annualize
mus <- colMeans(market_linret)*N
sigs <- apply(market_linret, 2, sd)*sqrt(N)
data.frame(mus, sigs)

# calculate sample covariance matrix of daily linear returns then annualized
Sig <- cov(market_linret)*N

# correlation matrix
cor(market_linret)
Caveat

This way of calculating the expected returns and the covariance matrix of the market at investment horizon is doubtful and questionable.
In [7]: max(Sig)  
    min(Sig)  

Out[7]: 0.0674507844416095  
Out[7]: 0.0131190028569719  

In [8]: # create a vector of ones  
    u <- rep(1, dim(market)[2])  
    # find the inverse of covariance matrix Sig  
    Sig_inv <- solve(Sig)  
    # determine the weights of minimum variance portfolio  
    w_mvp <- Sig_inv %*% u  
    w_mvp <- w_mvp/sum(w_mvp)  
    w_mvp  
    # mean and variance of minimum variance portfolio  
    mu_mvp <- t(w_mvp) %*% mus  
    sig_mvp <- sqrt(t(w_mvp) %*% Sig %*% w_mvp)  
    data.frame(mu_mvp, sig_mvp)  

Out[8]:  
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<th>sig_mvp</th>
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Out[8]:  
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</tr>
</thead>
<tbody>
<tr>
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<td>0.14216530</td>
</tr>
</tbody>
</table>
# weights of portfolios on efficient frontier

```r
eff_front <- function(mu){
  A <- as.numeric(u%*%Sig_inv%*%mus)
  B <- as.numeric(mus%*%Sig_inv%*%mus)
  C <- as.numeric(u%*%Sig_inv%*%u)

  numer <- B*Sig_inv%*%u - A*Sig_inv%*%mus + mu*(C*Sig_inv%*%mus - A*Sig_inv%*%u)
  denom <- B*C - A^2
  weights <- numer/denom
  sig <- sqrt((B - 2*mu*A + mu^2*C)/denom)

  list(weights=weights, sig=sig)
}

eff_front(0.05)
```

```
# rescale plot
options(repr.plot.width=6, repr.plot.height=4.5)
```
```r
In [12]:

# vectorize the eff_front function
tmp <- function(x) {sapply(x, function(t) eff_front(t)$sig)}

# plot efficient frontier (with short selling)
y <- seq(as.numeric(mu_mvp), 0.4, by=0.01)
x <- tmp(y)
plot(x, y, type='l', xlab=expression(sigma), ylab=expression(mu), ylim=c(-0.4, max(y)))

# add minimum variance portfolio to the figure
points(sig_mvp, mu_mvp, col='blue')
text(sig_mvp+0.015, mu_mvp-0.015, labels='mvp')

# add individual stocks to the figure
points(sigs, mus, col='red')
names <- c('aapl', 'goog', 'axp', 'xom', 'gm', 'pfe')
text(sigs+0.015, mus-0.015, labels=names)

# add lower branch
yl <- seq(-0.5, as.numeric(mu_mvp), by=0.01)
xl <- tmp(yl)
lines(xl, yl, lty=2)
abline(h=0, lty=2, col='green')
```
In [15]:
# set the risk free rate
rf <- 0.01

# weights of market portfolio
w_M <- Sig_inv%*%(mus - rf*u)
w_M <- w_M/sum(w_M)
w_M

Out[15]:
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL.Adjusted</td>
<td>0.3198358</td>
</tr>
<tr>
<td>GOOG.Adjusted</td>
<td>0.7717506</td>
</tr>
<tr>
<td>AXP.Adjusted</td>
<td>0.09326069</td>
</tr>
<tr>
<td>XOM.Adjusted</td>
<td>-0.8780114</td>
</tr>
<tr>
<td>GM.Adjusted</td>
<td>-0.1047212</td>
</tr>
<tr>
<td>PFE.Adjusted</td>
<td>0.7978855</td>
</tr>
</tbody>
</table>
# expected return and standard deviation of the market portfolio
mu_M <- t(w_M) %*% mus
sig_M <- sqrt(t(w_M) %*% Sig %*% w_M)
data.frame(mu_M, sig_M)

# plot efficient frontier (with short selling)
y <- seq(as.numeric(mu_mvp), 0.6, by=0.01)
x <- tmp(y)
plot(x, y, type='l', xlab=expression(sigma), ylab=expression(mu), ylim=c(-0.2, max(y)))

# add minimum variance portfolio
points(sig_mvp, mu_mvp, col='blue')
text(sig_mvp+0.015, mu_mvp-0.015, labels='mvp')

# add individual stocks
points(sigs, mus, col='red')
names <- c('aapl', 'goog', 'axp', 'xom', 'gm', 'pfe')
text(sigs+0.015, mus-0.015, labels=names)

# add lower branch
yl <- seq(-0.5, as.numeric(mu_mvp), by=0.01)
xl <- tmp(yl)
lines(xl, yl, lty=2)

# add market portfolio
points(sig_M, mu_M, col='orange')
text(sig_M + 0.015, mu_M - 0.015, labels='M')

# add capital market line
abline(a=rf, b=(mu_M-rf)/sig_M, col='blue')

Out[16]:

<table>
<thead>
<tr>
<th>mu_M</th>
<th>sig_M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.302095</td>
</tr>
</tbody>
</table>
# linear return series of market portfolio

```
In [17]:
r_M <- apply(market_linret, 1, function(x) {sum(x*w_M)})

cov(market_linret$AAPL.Adjusted, r_M)
apply(market_linret, 2, function(x) {cov(x, r_M)})
```

```
Out[17]: 0.000160967488941467

Out[17]:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL.Adjusted</td>
<td>0.000160967488941467</td>
</tr>
<tr>
<td>GOOG.Adjusted</td>
<td>0.000207948658499625</td>
</tr>
<tr>
<td>AXP.Adjusted</td>
<td>8.34193577556428e-05</td>
</tr>
<tr>
<td>XOM.Adjusted</td>
<td>-1.67116620443415e-05</td>
</tr>
<tr>
<td>GM.Adjusted</td>
<td>6.232559299616024e-05</td>
</tr>
<tr>
<td>PFE.Adjusted</td>
<td>0.000117395942943619</td>
</tr>
</tbody>
</table>
```

```
In [18]:
betas <- apply(market_linret, 2, function(x) {cov(x, r_M)*N/sig_M^2})

betas
```

```
Out[18]:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL.Adjusted</td>
<td>0.500579703027288</td>
</tr>
<tr>
<td>GOOG.Adjusted</td>
<td>0.646882621448592</td>
</tr>
<tr>
<td>AXP.Adjusted</td>
<td>0.259419076526884</td>
</tr>
<tr>
<td>XOM.Adjusted</td>
<td>-0.051970238700132</td>
</tr>
<tr>
<td>GM.Adjusted</td>
<td>0.193821292864073</td>
</tr>
<tr>
<td>PFE.Adjusted</td>
<td>0.365080095625364</td>
</tr>
</tbody>
</table>
```

```
In [18]:

head(r_M)

www <- c(1,0,0,0,0,0)

xxx <- apply(market_linret, 1, function(x) {sum(x*www)})

head(xxx)
```

```
2015-01-05       -0.00376024032044283
2015-01-06       -0.00286776391797913
2015-01-07       0.00480081806303296
2015-01-08       0.0141733314177827
2015-01-09       -0.00524884768315583
2015-01-12       0.00854652080492492

2015-01-05       -0.0281715548344598
2015-01-06       9.40223933225681e-05
2015-01-07       0.014022274714332
2015-01-08       0.038422326970837
2015-01-09       0.0017246478074163
2015-01-12       -0.0246406198528519
```
In [24]:
fit <- lm(market_linret$AAPL.Adjusted ~ r_M)
fit
# summary(fit)
fit$coeff
data.frame(fit$coeff[1], fit$coeff[2])

Out[24]:
Call:
  lm(formula = market_linret$AAPL.Adjusted ~ r_M)
Coefficients:
  (Intercept)          r_M
   1.982e-05    5.006e-01

Out[24]:
   (Intercept)  1.98182657528852e-05
 r_M           0.500579703027288

Out[24]:
   fit.coe1.       fit.coe2.
(Intercept)   1.981827e-05   0.5005797

In [25]:
# verify beta by regression
lm(market_linret$AAPL.Adjusted ~ r_M)$coeff
lm(market_linret$GOOG.Adjusted ~ r_M)$coeff
#summary(lm(xxx ~ r_M))

Out[25]:
   (Intercept)  1.98182657528852e-05
 r_M           0.500579703027288

Out[25]:
   (Intercept)  1.40205308948972e-05
 r_M           0.646682621448591
In [20]:

# verify the security market line
mus - rf
as.numeric(mu_M - rf)*betas

# difference is basically zero
mus - rf - (as.numeric(mu_M - rf)*betas)

AAPL.Adjusted 0.146729095191831
GOOG.Adjusted 0.19960268682234
AXP.Adjusted 0.0597557050913836
XOM.Adjusted -0.00804260380533965
GM.Adjusted 0.093328779713846
PFE.Adjusted 0.113081588145057

AAPL.Adjusted 0.156729095191831
GOOG.Adjusted 0.20960268682234
AXP.Adjusted 0.0697557050913836
XOM.Adjusted 0.00195739619466034
GM.Adjusted 0.103382577971385
PFE.Adjusted 0.123081588145057

In [26]:

# load required package quadprog
library(quadprog)

# the function solve.QP solves the constrained quadratic minimization problem
# min 1/2 x'Dx - d'x
# subject to
# A' x >= b
# inputs of the function solve.QP are
# Dmat, dvec, Amat, bvec, meq
# meq indicates the first meq constraints are equalities, the rest are inequalities

Warning message:
: package 'quadprog' was built under R version 3.2.3

In [22]: mu
In [38]:
# portfolio on the efficient frontier without short selling
w_eff_shorsel <- function(mu) {
  N <- 6 # number of risky assets in the example
  Dmat <- 2*Sig
  dvec <- 0*(1:N)
  Amat <- cbind(mus, u, diag(N)) # the function diag(n) creates an n by n identity matrix
  bvec <- c(mu, 1, 0*(1:N))
  solve.QP(Dmat, dvec, Amat, bvec, meq=2)
}

w_eff_shorsel(0.02)

Out[38]:
$solution
 0 -3.44773753253103e-17 0.146728188969125 0.722506713492282 0.0818617436134165
 0.0489033539251762
$value
0.0268337276915023
$unconstrained.solution
 0 0 0 0 0 0
$niterations
5 0
$lagrangian
0.255010104924806 0.0587676574815006 0.015074364771773 0.0246151390281826 0
0 0 0
$name
2 1 4 3

In [36]:
# efficient frontier
yy <- seq(min(mus)+0.001, max(mus)-0.001, by=0.001)
x <- sapply(yy, function(x) {sqrt(w_eff_shorsel(x)$value)})
```r
In [37]:
y <- seq(as.numeric(mu_mvp), 0.35, by=0.01)
x <- tmp(y)
plot(x, y, type='l', xlab=expression(sigma), ylab=expression(mu), ylim=c(-0.2, max(y)))

# add minimum variance portfolio
points(sig_mvp, mu_mvp, col='blue')
text(sig_mvp+0.0055, mu_mvp-0.0055, labels='mvp')

# add individual stocks
points(sigs, mus, col='red')
names <- c('aapl', 'goog', 'axp', 'xom', 'gm', 'pfe')
text(sigs+0.0055, mus-0.0055, labels=names)

# add lower branch
yl <- seq(-0.5, as.numeric(mu_mvp), by=0.01)
xl <- tmp(yl)
lines(xl, yl, lty=2)

# plot portfolios without short selling
points(xx, yy, col='violet', type='l')
```

In [26]:
```
# download SPX from Yahoo Finance
getSymbols('^GSPC', from='2015-01-01')
```

'GSPC'
In [27]:

```r
spx <- GSPC$GSPC.Adjusted

# calculate the daily log return of spx
spx_logret <- diff(log(spx))[-1] # remove the first NA
head(spx_logret)

# transform daily log return to daily linear return
spx_linret <- exp(spx_logret) - 1
head(spx_linret)
```

```
GSPC.Adjusted
2015-01-05  -0.018447213
2015-01-06  -0.008933255
2015-01-07   0.011562736
2015-01-08   0.017730169
2015-01-09  -0.008439322
2015-01-12  -0.008126617
```

In [1]:

```r
# Calculates the beta of the target portfolio by linear regression (OLS)

lm_fit <- lm(market_linret$AAPL.Adjusted ~ spx_linret)
summary(lm_fit)
```

```
Error in eval(expr, envir, enclos): object 'market_linret' not found

Error in summary(lm_fit): object 'lm_fit' not found
```
Problems with CAPM


In their 2004 review, economists Eugene Fama and Kenneth French argue that "the failure of the CAPM in empirical tests implies that most applications of the model are invalid".

- The traditional CAPM using historical data as the inputs to solve for a future return of asset $i$. However, the history may not be sufficient to use for predicting the future and modern CAPM approaches have used betas that rely on future risk estimates.
- Most practitioners and academics agree that risk is of a varying nature (non-constant). A critique of the traditional CAPM is that the risk measure used remains constant (non-varying beta). Recent research has empirically tested time-varying betas to improve the forecast accuracy of the CAPM.
- The model assumes that the variance of returns is an adequate measurement of risk. This would be implied by the assumption that returns are normally distributed, or indeed are distributed in any two-parameter way, but for general return distributions other risk measures (like coherent risk measures) will reflect the active and potential shareholders' preferences more adequately. Indeed, risk in financial investments is not variance in itself, rather it is the probability of losing: it is asymmetric in nature. Barclays Wealth have published some research on asset allocation with non-normal returns which shows that investors with very low risk tolerances should hold more cash than CAPM suggests.
- The model does not appear to adequately explain the variation in stock returns. Empirical studies show that low beta stocks may offer higher returns than the model would predict. Some data to this effect was presented as early as a 1969 conference in Buffalo, New York in a paper by Fischer Black, Michael Jensen, and Myron Scholes. Either that fact is itself rational (which saves the efficient-market hypothesis but makes CAPM wrong), or it is irrational (which saves CAPM, but makes the EMH wrong – indeed, this possibility makes volatility arbitrage a strategy for reliably beating the market).
- The model assumes that there are no taxes or transaction costs, although this assumption may be relaxed with more complicated versions of the model.
- The market portfolio should in theory include all types of assets that are held by anyone as an investment (including works of art, real estate, human capital...) In practice, such a market portfolio is unobservable and people usually substitute a stock index as a proxy for the true market portfolio. Unfortunately, it has been shown that this substitution is not innocuous and can lead to false inferences as to the validity of the CAPM, and it has been said that due to the inobservability of the true market portfolio, the CAPM might not be empirically testable. This was presented in greater depth in a paper by Richard Roll in 1977, and is generally referred to as Roll's critique.
- The model assumes just two dates, so that there is no opportunity to consume and rebalance portfolios repeatedly over time. The basic insights of the model are extended and generalized in the intertemporal CAPM (ICAPM) of Robert Merton, and the consumption CAPM (CCAPM) of Douglas Breeden and Mark Rubinstein.
- Empirical tests show market anomalies like the size and value effect that cannot be explained by the CAPM. For details see the Fama–French three-factor model.
Factor models

CAPM is a single factor model in which the market portfolio is regarded as the only “factor”.

It is tempting to generalize the CAPM to multifactors:

\[ r_i = \alpha + \sum_j \beta_{ij} f_j + \epsilon_i \]

where the \( \beta_{ij} \)'s are referred to as the factor loadings (or simply the loadings for short).

- The choice of factors and their corresponding loadings are more like an art than science. Typical choices are macroeconomics indices such as the Consumer Pricing Index (CPI), employment rate, etc.
- Commonly used models include
  - Time series models, in which factors are specified and loadings are inferred from the data
  - Cross sectional models, in which loadings are specified, the factors are estimated
  - Statistical models, in which both loadings and factors are to be inferred from the data

Arbitrage pricing theory (APT)

From this page (https://www.investopedia.com/terms/a/apt.asp) in Investopedia:

- Arbitrage pricing theory (APT) is a multi-factor asset pricing model based on the idea that an asset's returns can be predicted using the linear relationship between the asset's expected return and a number of macroeconomic variables that capture systematic risk.
- Unlike the CAPM, which assume markets are perfectly efficient, APT assumes markets sometimes misprice securities, before the market eventually corrects and securities move back to fair value.
- Using APT, arbitrageurs hope to take advantage of any deviations from fair market value.
- APT factors are the systematic risk that cannot be reduced by the diversification of an investment portfolio. The macroeconomic factors that have proven most reliable as price predictors include unexpected changes in inflation, gross national product (GNP), corporate bond spreads and shifts in the yield curve. Other commonly used factors are gross domestic product (GDP), commodities prices, market indices and exchange rates.

Consider an economy consists of \( n \) risky assets with (random) returns \( r_i \), for \( i = 1, \cdots, n \), and a risk free asset with rate \( r_f \).

Arbitrage pricing theory assumes the multifactor model for returns of the risk assets

\[ r_i = \alpha_i + \sum_j \beta_{ij} f_j + \epsilon_i, \text{ for } 1 \leq i \leq n \]

- the factor \( f_j \)'s are considered as the systematic risks
- the \( \epsilon_i \)'s are assumed mean zero, uncorrelated with each other and with the factors \( f_j \). They are regarded as the idiosyncratic risks.

APT asserts that the expected returns \( \ell \) satisfies

\[ \ell_i = \mathbb{E} [r_i] = r_f + \sum_j \beta_{ij} \mathbb{E} [f_j] \]

for \( i = 1, \cdots, n \).

However, the APT itself does not specify how to select the factors.
Fama-French 3-factor model
Fama and French observed that two classes of stocks tend to do better than the market as a whole: (i) small caps and (ii) stocks with a high book-to-market ratio (B/P, customarily called value stocks, contrasted with growth stocks).
They added two factors to CAPM to reflect a portfolio’s exposure to these two classes
$$\mu - r_f = \alpha + \beta_M (\mu_M - r_f) + \beta_{SMB} \cdot SMB + \beta_{HML} \cdot HML$$
- $\mu$: portfolio’s expected return
- $r_f$: risk free rate
- SMB stands for “Small Minus Big” in market capitalization
- HML for “High Minus Low” in book-to-market ratio; they measure the historic excess returns of small caps over big caps and of value stocks over growth stocks.

Note
- To implement the model, we further need the data for SMB and HML.

Black-Litterman model
Rather than taking the expected returns $\mathbf{\mu}$ and the covariance matrix $\mathbf{\Sigma}$ of the market as given (let it be estimated from the historical data or from any other more sophisticated statistical model), the Black-Litterman model allows the investor (or a fund manager) to blend in his own take on the numbers, called views, into the process of portfolio optimization in a Bayesian statistics style. The output the Black-Litterman model is an optimal portfolio weights adjusted to the investor’s own take.

Other return-risk trade-off maximization
The Markowitz model of portfolio theory can be recast as
$$\max_w w' \mathbf{\mu} - \frac{1}{2} w' \mathbf{\Sigma} w$$
subject to the constraint $w' \mathbf{\mu} = 1$. In this setting, we are penalizing the expected return by a factor of variance. Rather than using variance as penalty, one may consider using other risk measures for penalty, which in turn ends up optimization problems such as
- mean-VaR optimization
- mean-risk measure optimization
- Utility maximization

Note
Under the assumption of normal market, that is, the linear returns are jointly normally distributed, the above problems are all equivalent to the mean-variance optimization.

Portfolio optimization in continuous time
HARA utility

- HARA stands for Hyperbolic Absolute Risk Aversion.
  - The Arrow-Pratt measure of absolute risk-aversion (ARA), a.k.a., the coefficient of absolute risk aversion, $A$ of a utility function $U$ is defined as $A = -\frac{U''}{U'}$.
  - The reciprocal of $A$ is referred to as risk tolerance $T = \frac{1}{A}$.
  - A utility function is said to exhibit hyperbolic absolute risk aversion if and only if the risk tolerance $T$ is a linear function:
    $$T(x) = \frac{x}{1 - \gamma} + \frac{b}{a}.$$
  - HARA utility functions have the general form
    $$U(x) = \frac{1 - \gamma}{\gamma} \left( \frac{ax}{1 - \gamma} + b \right)^\gamma.$$
  - HARA utility is often quoted as a power utility $U(x) = x^\gamma$, for $\gamma \in [0, 1]$.

Note

- Another commonly used utility is CARA (Constant Absolute Risk Aversion) which results in an exponential utility.

The economy

The economy consists of one risky asset and one risk free asset.

- $M_t$: price of risk free asset
- $S_t$: price of risky asset
- $u_t$: percentage/weight of wealth invested in the risk asset at time $t$
- $W_t$: investor’s wealth at time $t$

Evolution of the assets

Let $Z_t$ be a Brownian motion.

- Riskless asset follows
  $$dM_t = rM_t dt, \quad M_0 = 1$$
- Risky asset follows a geometric Brownian motion
  $$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$
  - $r, \mu, \text{ and } \sigma > 0$ are constants. Assume $\mu > r > 0$.

Self-financing strategy

Let $h_t$ be the number of shares of the risky asset in the investor’s portfolio and $b_t$ dollars in the riskless assets. Apparently, the investor’s wealth $W_t$ at time $t$ is given by
  $$W_t = h_t S_t + b_t.$$ 

A trading strategy $(h_t, b_t)$ is called self-financing if the wealth process satisfies
  $$dW_t = h_t dS_t + rb_t dt.$$
Wealth process

We rewrite the wealth in terms of $u_t$ as

$$W_t = u_t W_t + (1 - u_t) W_t = \frac{u_t W_t}{S_t} S_t + (1 - u_t) W_t.$$  

The condition of self-financing implies that $W_t$ satisfies

$$dW_t = \frac{u_t W_t}{S_t} dS_t + (1 - u_t) W_t r dt$$

or equivalently

$$\frac{dW_t}{W_t} = u_t (\mu dt + \sigma dZ_t) + (1 - u_t) r dt$$  

$$= \{r + u_t (\mu - r)\} dt + u_t \sigma dZ_t$$

Merton's portfolio problem

Merton's problem is recast as the following stochastic control problem

$$\max_{u \in A} \mathbb{E} \left[ U(W_T) \right],$$

where $W_t$ follows the controlled geometric Brownian motion

$$\frac{dW_t}{W_t} = \{r + u_t (\mu - r)\} dt + u_t \sigma dZ_t.$$  

Note

- $W_t \geq 0$

Optimal control theory


Optimal control theory, an extension of the calculus of variations, is a mathematical optimization method for deriving control policies. The method is largely due to the work of Lev Pontryagin and his collaborators in the Soviet Union and Richard Bellman in the United States.

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional. The optimal control can be derived using Pontryagin's maximum principle (a necessary condition also known as Pontryagin's minimum principle or simply Pontryagin's Principle), or by solving the Hamilton–Jacobi–Bellman equation (a sufficient condition).
Richard Ernest Bellman

Courtesy: Photos from Wikipedia

 Levinovich Pontryagin

Courtesy: Photo from Wikipedia

Bellman’s principle of optimality

“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”

(See Bellman, 1957, Chap. III.3.)

The value function

The value function is defined as

\[
V(t, w) = \max_{u \in A(t, T)} \mathbb{E} \left[ U(W_T) \mid W_t = w \right].
\]
Hamilton-Jacobi-Bellman (HJB) equation

The value function $V(t, w)$ satisfies the HJB equation

$$V_t + \max_u \left\{ \frac{\sigma^2 u^2}{2} w^2 V_{ww} + ((\mu - r)u + r) w V_w \right\} = 0$$

with terminal condition $V(w) = U(w)$.

Note

For general utility function $U$, it is hard, if not impossible, to find closed form solution to the above HJB equation except the HARA utility.

First order criterion

Assuming $V_{ww} < 0$.

$$u = -\frac{(\mu - r)wV_w}{\sigma^2 w^2 V_{ww}} = -\frac{(\mu - r)V_w}{\sigma^2 V_{ww}}$$

The HJB equation becomes

$$V_t - \frac{(\mu - r)^2 V_w^2}{2\sigma^2 V_{ww}} + rwV_w = 0$$

Power ansatz

Suppose the value function $V$ is of the form

$$V(t, w) = a(t)w^\gamma$$

Plug the ansatz into the HJB equation yields

$$a'(t)w^\gamma + \frac{(\mu - r)^2}{2\sigma^2} \frac{\gamma}{1 - \gamma} a(t)w^\gamma + r\gamma a(t)w^\gamma = 0$$

Thus, $a$ satisfies

$$a'(t) + \left\{ \frac{(\mu - r)^2}{2\sigma^2} \frac{\gamma}{1 - \gamma} + r\gamma \right\} a(t) = 0$$

with terminal condition $a(T) = 1$.

$a$ has the closed form expression

$$a(t) = e^{\alpha(T-t)}$$

where $\alpha = \frac{(\mu - r)^2}{2\sigma^2} \frac{\gamma}{1 - \gamma} + r\gamma$. Hence, the value function $V$ is given by

$$V(t, w) = e^{\alpha(T-t)} w^\gamma.$$ 

Note

- $V_{ww} = e^{\alpha(T-t)} \gamma(\gamma - 1)w^{\gamma-2} < 0$ since $\gamma \in (0, 1)$.
Optimal weight - the Merton ratio

The optimal strategy $u^*$ reads

$$u^* = \frac{\mu - r}{\sigma^2(1 - \gamma)}$$

In other words, the optimal strategy is to maintain a constant percentage/weight $u^*$, determined by the Merton ratio, of the wealth invested in the risky asset.

Note

- The higher the excess return $\mu - r$ of the risky asset, the higher the weight $u^*$.
- The higher the volatility $\sigma$ of the risky asset, the lower the weight $u^*$.
- The less risk averse the investor is, i.e., $\gamma$ is closer to 1, the higher the weight $u^*$. 