Probability density of lognormal fractional SABR model

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Outline

- Review of SABR model and SABR formula
- Lognormal fractional SABR (fSABR) model
- A bridge representation for probability density of lognormal fSABR
- Edgeworth type of expansion
- Heuristic derivation of sample path large deviation principle from path-integral perspective
- Approximations of implied volatility in small time
- Conclusion
Stochastic $\alpha \beta \rho$ (SABR) model

Stochastic $\alpha \beta \rho$ (SABR) model was suggested and investigated by Hagan-Lesniewski-Woodward as

$$dS_t = S_t^\beta \alpha_t (\rho dB_t + \bar{\rho} dW_t), \quad S_0 = s;$$

$$d\alpha_t = \nu \alpha_t dB_t, \quad \alpha_0 = \alpha$$

where $B_t$ and $W_t$ are independent Brownian motions, $\bar{\rho} = \sqrt{1 - \rho^2}$.

- SABR model is market standard for quoting cap and swaption volatilities using the SABR formula for implied volatility. Nowadays also used in FX and equity markets.
- $\beta = 0$ is referred to as normal SABR
- $\beta = 1$ is referred to as lognormal SABR
SABR formula

The SABR formula is a small time asymptotic expansion up to first order for the implied volatilities of call/put option induced by the SABR model.

$$\sigma_{BS}(K, \tau) = \nu \frac{\log(s/K)}{D(\zeta)} \{1 + O(\tau)\}$$

as the time to expiry $\tau$ approaches 0. $D$ and $\zeta$ are defined respectively as

$$D(\zeta) = \log \left( \frac{\sqrt{1 - 2\rho \zeta + \zeta^2 + \zeta - \rho}}{1 - \rho} \right)$$

and

$$\zeta = \left\{ \begin{array}{ll} \frac{\nu}{\alpha} \frac{s^{1-\beta} - K^{1-\beta}}{1-\beta} & \text{if } \beta \neq 1; \\ \frac{\nu}{\alpha} \log \left( \frac{s}{K} \right) & \text{if } \beta = 1. \end{array} \right.$$
Plot of SABR formula for implied volatilities

\[ \alpha = 0.13927, \quad \rho = -0.06867, \quad \nu = 0.5778, \quad \tau = 1. \]
Brief review on derivation of SABR formula

Take \( \nu = 1 \) for simplicity. Change of variable

\[
x = \frac{1}{\bar{\rho}} \left( \int_{s_0} ds \frac{ds}{s^\beta} - \rho \alpha \right)
\]

\[
y = \begin{cases} 
\frac{1}{\bar{\rho}} \left( \frac{s^{1-\beta} - s_0^{1-\beta}}{1-\beta} \right) - \frac{\rho}{\bar{\rho}} \alpha & \text{if } \beta \neq 1; \\
\frac{1}{\bar{\rho}} \log \left( \frac{s}{s_0} \right) - \frac{\rho}{\bar{\rho}} \alpha & \text{if } \beta = 1,
\end{cases}
\]

\[
y = \alpha
\]
Ito’s formula implies

\[ dX_t = Y_t dW_t - \frac{\beta}{2\bar{\rho} (1 - \beta)} \frac{Y_t^2}{(\bar{\rho} X_t - \rho Y_t)} dt \]

\[ dY_t = Y_t dB_t \]

Infinitesimal generator

\[ \mathcal{L} = y^2 (\partial_x^2 + \partial_y^2) - \frac{\beta}{2\bar{\rho} (1 - \beta)} \frac{y^2}{(\bar{\rho} x - \rho y)} \partial_x \]

- The principle part is the Laplace-Beltrami operator on 2-dimensional hyperbolic space. The corresponding diffusion process is called the \textit{hyperbolic Brownian motion}.
- The transition density of 2-dimensional hyperbolic Brownian motion is given by the McKean kernel.
- Drift part does not play a role in the large deviation regime.
SABR density in small time

The transition density of the process \((X_t, Y_t)\) from \((x, y)\) to \((\xi, \eta)\) in time \(\tau\) is asymptotically given by

\[
p_\tau(\xi, \eta|x, y) \approx \frac{1}{2\pi\tau} e^{-\frac{d^2(\xi, \eta|x, y)}{2\tau}},
\]

where \(d\) is defined as

\[
d(\xi, \eta|x, y) = \cosh^{-1} \left[ \frac{(\xi - x)^2 + \eta^2 + y^2}{2y\eta} \right]
\]

with

\[
\cosh^{-1} z = \log \left( z + \sqrt{z^2 - 1} \right).
\]

\(d\) is the geodesic distance between the initial point \((x, y)\) and terminal point \((\xi, \eta)\).
From density to option price

For out-of-money calls,

\[ C(K, T) = \int\int (s_T - K)^+ p_T(s_T, \alpha_T|s_0, \alpha_0)ds_T d\alpha_T \]

\[ \approx \frac{1}{2\pi T} \int_0^\infty \int_K^\infty (s_T - K)e^{-\frac{d^2(s_T, \alpha_T|s_0, \alpha_0)}{2T}} ds_T d\alpha_T \]

By applying the Laplace asymptotic formula, we have, as \( T \to 0 \),

\[ C(K, T) \approx e^{-\frac{d^2(s_0, \alpha_0)}{2T}} \]

where \( d_* \) is the minimal distance from the initial point \((s_0, \alpha_0)\) to the half plane \( \{(s, \alpha) : s \geq K\} \), i.e.,

\[ d_*(s_0, \alpha_0) = \min \left\{ d(s, \alpha|s_0, \alpha_0) : s \geq K \right\}. \]
Matching with Black-Scholes price

The zeroth order SABR formula is thus obtained by matching the exponent with the corresponding term in Black-Scholes price.

\[ e^{-\frac{d_*^2(s_0, \alpha_0)}{2T}} \approx C(K, T) = C_{BS}(K, T) \approx e^{-\frac{(\log s_0 - \log K)^2}{2\sigma_{BS}^2 T}} \]

We end up with

\[ d_*^2(s_0, \alpha_0) \approx \frac{(\log s_0 - \log K)^2}{\sigma_{BS}^2} \]

Thus,

\[ \sigma_{BS}(K, T) \approx \frac{|\log s_0 - \log K|}{d_*(s_0, \alpha_0)}. \]
Why fractional process?

Gatheral-Jaisson-Rosenbaum observed from empirical data that

- Log-volatility behaves as a fractional Brownian Motion with Hurst exponent $H$ of order 0.1 at any reasonable time scale. Indeed, they fitted the empirical $q$th moments $m(q, \Delta)$ in various lags $\Delta$ to

$$\mathbb{E} [ | \log \sigma_{t+\Delta} - \log \sigma_t |^q ] = K_q \Delta^{\zeta_q}$$

proxied by daily realized variance estimates. $K_q$ denotes the $q$th moment of standard normal.

- At-the-money volatility skew is well approximated by a power law function of time to expiry
Gatheral-Jaisson-Rosenbaum

Log-volatility behaves as a fractional Brownian Motion with Hurst exponent $H$ of order 0.1 at any reasonable time scale.
Gatheral-Jaisson-Rosenbaum

Log-log plot of $m(q, \Delta)$ versus $\Delta$ for various $q$. 
Gatheral-Jaisson-Rosenbaum

At-the-money volatility skew \( \psi(\tau) = \left| \frac{d}{dk} \right|_{k=0} \sigma_{BS}(k, \tau) \) is well approximated by a power law function of time to expiry \( \tau \).

Figure 1.2: The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013; the red curve is the power-law fit \( \psi(\tau) = A \tau^{-0.4} \).
The observations suggest the following model for instantaneous volatility

\[ \sigma_t = \sigma_0 e^{\nu W_t^H}, \]

where \( W^H \) is a fractional Brownian motion with Hurst exponent \( H \). As stationarity of \( \sigma_t \) is concerned, GJR suggested the model for instantaneous volatility as \( \sigma_t = \sigma_0 e^{X_t} \) where

\[ dX_t = \alpha(m - X_t)dt + \nu dW_t^H \]

is a fractional Ornstein-Uhlenbeck process. Again, drift term plays no role in large deviation regime.
Review: fractional Brownian motion

A mean-zero Gaussian process $B_t^H$ is called a fractional Brownian motion with Hurst exponent $H \in [0, 1]$ if its autocovariance function $R(t, s)$, for $t, s > 0$, satisfies

$$R(t, s) := \mathbb{E} \left[ B_t^H B_s^H \right] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).$$

- $B_t^H$ is self-similar, indeed, $B_{at}^H \overset{d}{=} a^H B_t^H$ for $a > 0$
- $B_t^H$ has stationary increments
- $B_t^H$ is a standard Brownian motion when $H = \frac{1}{2}$
Lognormal fSABR model

Consider the following lognormal fSABR model

$$\frac{dS_t}{S_t} = \alpha_t (\rho dB_t + \bar{\rho} dW_t),$$

$$\alpha_t = \alpha_0 e^{\nu B_t^H},$$

where $B_t$ and $W_t$ are independent Brownian motions, $\bar{\rho} = \sqrt{1 - \rho^2}$. $B_t^H$ is a fractional Brownian motion with Hurst exponent $H$ driven by $B_t$:

$$B_t^H = \int_0^t K_H(t, s) dB_s.$$

$K_H$ is the Molchan-Golosov kernel.

- Goal: to obtain an easy to access expression for the joint density of $(S_t, \alpha_t)$. 
Slightly more explicit form

Defining the new variables $X_t = \log S_t$ and $Y_t = \alpha_t$, we may rewrite the lognormal fSABR model in a slightly more explicit form as

$$X_t - X_0 = Y_0 \int_0^t e^{\nu B_s^H} \left( \rho dB_s + \bar{\rho} dW_s \right) - \frac{Y_0^2}{2} \int_0^t e^{2\nu B_s^H} ds,$$

$$Y_t = Y_0 e^{\nu B_t^H}.$$

- We derive a bridge representation for the joint density of $(X_t, Y_t)$ in a “Fourier space”.
The joint density of \((X_t, Y_t)\) has the following bridge representation

\[
p(t, x_t, y_t | x_0, y_0) = \frac{e^{-\frac{\eta^2_{t}}{2\nu^2 t^{2H}}}}{y_t \sqrt{2\pi \nu^2 t^{2H}}} \times \frac{1}{2\pi} \times \int e^{i(x_t-x_0)\xi} \mathbb{E}\left[ e^{i\left(-\rho \int_0^t y_0 e^{\nu B^H_s} dB_s + \frac{y_0^2}{2} v_t\right)\xi} e^{-\frac{\rho^2 y_0^2 v_t}{2} \xi^2} \left| \nu B^H_t = \eta_t \right\} \right] d\xi,
\]

where \(i = \sqrt{-1}\), \(v_t = \int_0^t e^{2\nu B^H_s} ds\) and \(\eta_t = \log \frac{y_t}{y_0}\).
Bridge representation in uncorrelated case

The bridge representation for the joint density of \((X_t, Y_t)\) reads simpler when \(\rho = 0\):

\[
p(t, x_t, y_t | x_0, y_0) = \frac{e^{-\frac{\eta^2_t}{2\nu^2 t^{2H}}}}{y_t \sqrt{2\pi \nu^2 t^{2H}}} \times \frac{1}{2\pi} \int \frac{e^{i(x_t - x_0)\xi}}{\mathbb{E} \left[ e^{-\frac{1}{2}(\xi - i)\xi y_0^2 v_t | \nu B^H_t = \eta_t} \right]} d\xi,
\]

where \(i = \sqrt{-1}, v_t = \int_0^t e^{2\nu B^H_s} ds\) and \(\eta_t = \log \frac{y_t}{y_0}\).
The McKean kernel $p_{H^2}(t, x_t, y_t|x_0, y_0)$ reads

$$p_{H^2}(t, x_t, y_t|x_0, y_0) = \frac{\sqrt{2} e^{-t/8}}{(2\pi t)^{3/2}} \int_0^\infty \frac{\xi e^{-\xi^2/2t}}{\sqrt{\cosh \xi - \cosh d}} d\xi,$$

where $d = d(x_t, y_t; x_0, y_0)$ is the geodesic distance from $(x_t, y_t)$ to $(x_0, y_0)$.

- Note that the McKean kernel is a density with respect to the Riemannian volume form $\frac{1}{y_t^2} dx_t dy_t$.
- The bridge representation can be regarded as a generalization of the McKean kernel.
- Indeed, in the case where $H = \frac{1}{2}$, $\nu = 1$ and $\rho = 0$, Ikeda-Matsumoto showed how to recover the McKean kernel.
Expanding around $b_s$

We expand the conditional expectation in the bridge representation around the deterministic path $b_s$. Let $\mathbb{E}_{\eta_t}[\cdot] = \mathbb{E}[\cdot | \nu B_t^H = \eta_t]$. First, define the deterministic path $b_s$ by

$$b_s = \log \mathbb{E}_{\eta_t} \left[ e^{2\nu B_s^H} \right].$$

Indeed,

$$b_s = \log \mathbb{E}_{\eta_t} [e^{2\nu B_s^H}] = 2\nu \mathbb{E}_{\eta_t} [B_s^H] + 2\nu^2 \text{var}_{\eta_t} [B_s^H]$$

$$= 2R(1, u)\eta_t + 2\nu^2 t^{2H} \left\{ u^{2H} - R^2(1, u) \right\},$$

where $u = \frac{s}{t}$ and $R(t, s) = \mathbb{E} [B_t^H B_s^H].$

- Note that $e^{b_s} = \mathbb{E}_{\eta_t} \left[ e^{2\nu B_s^H} \right]$. In other words, $e^{b_s}$ is an unbiased estimator of $e^{2\nu B_s^H}$ conditioned on $\nu B_t^H = \eta_t$. 
Now expand the conditional expectation in the bridge representation around the deterministic path $b_s$ as

$$
\mathbb{E}_{\eta_t} \left[ e^{-\frac{1}{2}(\xi-i)\xi} \int_0^t y_0^2 e^{2\nu B_s^H} ds \right] \\
= e^{-\frac{1}{2}(\xi-i)\xi} \int_0^t y_0^2 e^{bs} ds \mathbb{E}_{\eta_t} \left[ e^{-\frac{1}{2}(\xi-i)\xi} \int_0^t y_0^2 \left( e^{2\nu B_s^H} - e^{bs} \right) ds \right] \\
\approx e^{-\frac{1}{2}(\xi-i)\xi} \int_0^t y_0^2 e^{bs} ds \times \\
\left( 1 + \sum_{k=2}^n \frac{1}{k!} \mathbb{E}_{\eta_t} \left[ \left\{ \frac{1}{2}(\xi-i)\xi \int_0^t y_0^2 \left( e^{2\nu B_s^H} - e^{bs} \right) ds \right\}^k \right] \right).
$$

Note that, by definition of $b_s$, the first order term in the last expansion is automatically zero.
In principle each term in the expansion can be computed in closed form since they are given by the integrals of moments of shifted lognormal distributions. Denoting these terms by $H_k$, i.e.,

$$H_k(t, \eta_t) := \mathbb{E}_{\eta_t} \left[ \left\{ \int_0^t \left( e^{2\nu B^H_s} - e^{bs} \right) ds \right\}^k \right]$$

$$= \int\int_{[0,t]^k} \mathbb{E}_{\eta_t} \left[ \prod_{i=1}^k \left( e^{2\nu B^H_{s_i}} - e^{bs_i} \right) \right] ds_1 \cdots ds_k$$

For instance,

$$H_2(t, \eta_t) = \int\int_{[0,t]^2} \mathbb{E}_{\eta_t} \left[ \prod_{i=1}^2 \left( e^{2\nu B^H_{s_i}} - e^{2bs_i} \right) \right] ds_1 ds_2$$

$$= \int\int_{[0,t]^2} \mathbb{E}_{\eta_t} \left[ e^{2\nu B^H_{s_1} + 2\nu B^H_{s_2}} \right] ds_1 ds_2 - \hat{\nu}_t^2,$$

where $\hat{\nu}_t = \int_0^t e^{bs} ds$. 
Substituting the last expansion into bridge representation we obtain the following expansion (in the Fourier space) in terms of the $H_k$ functions as

$$p(t, x_t, y_t | x_0, y_0) \approx \frac{1}{y_t \sqrt{2\pi \nu^2 t^{2H}}} \times \frac{1}{2\pi} \int e^{i(x_t-x_0)\xi} e^{-\frac{1}{2}(\xi-i)\xi \hat{v}_t} \times \left\{ 1 + \sum_{k=2}^{n} \frac{1}{k!} \left[ -\frac{1}{2}(\xi - i)\xi \right]^k y_0^{2k} H_k(t, \eta_t) \right\} d\xi,$$

where $\hat{v}_t = \int_0^t y_0^2 e^{b_s} ds$.

- Note that it gives a natural expansion in $t$ via the $H_k$ functions.
Finally, integrating term by term, we obtain a full Edgeworth type of expansion for the probability density $p$ when $\rho = 0$ around the deterministic path $b_s$.

For example, we have up to second order

$$
\begin{align*}
p(t, x_t, y_t | x_0, y_0) & \approx e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}} \frac{1}{\sqrt{2\pi y_0^2 \hat{v}_t}} e^{-\frac{w_t^2}{2y_0^2 \hat{v}_t}} \times \\
& \left\{ 1 + \frac{1}{8} \left[ \frac{y_0^2}{\hat{v}_t}{\text{He}_2} \left( \frac{w_t}{\sqrt{\hat{v}_t}} \right) + \frac{2y_0}{\sqrt{\hat{v}_t}^3}{\text{He}_3} \left( \frac{w_t}{\sqrt{\hat{v}_t}} \right) + \frac{1}{\hat{v}_t^2}{\text{He}_4} \left( \frac{w_t}{\sqrt{\hat{v}_t}} \right) \right] y_0^4 H_2(t, \eta_t) \right\},
\end{align*}
$$

where $w_t = x_t - x_0 + \frac{y_0^2 \hat{v}_t}{2}$ and $\text{He}_n(\cdot)$ denotes the $n$th Hermite polynomial. Recall that $\hat{v}_t = \int_0^t e^{b_s} ds$ and $\eta_t = \log \frac{y_t}{y_0}$.
Small time asymptotics - uncorrelated

To the lowest order as $t \to 0$, the density $p$ has the following small time asymptotic behaviour

$$p(t, x_t, y_t | x_0, y_0) = \frac{e^{-\eta_t^2}}{y_t \sqrt{2\pi \nu^2 t^{2H}}} \frac{e^{-\frac{(x_t-x_0)^2}{2y_0^2 \hat{v}_t}}}{\sqrt{2\pi y_0^2 \hat{v}_t}} e^{\frac{x_t-x_0}{2}} \{1 + o(1)\},$$

where recall that $\hat{v}_t = \int_0^t e^{bs} ds$. 

Probability density in small time - correlated case

The Edgeworth type of expansion in the correlated case is more involved because of the appearance of the stochastic integral. However, the lowest order term is manageable. Define the functions $C_{RK}$ and $C_{eR}$ by

$$C_{RK}(\eta) := \int_0^1 e^{R(1,u)\eta} K_H(1,u) du, \quad C_{eR}(\eta) := \int_0^1 e^{2R(1,u)\eta} du.$$ 

Then to the lowest order we have

$$p(t, x_t, y_t | x_0, y_0) \approx \frac{1}{2\pi} \times \frac{1}{y_t \sqrt{\nu^2 t^{2H}}} e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}} \times \frac{1}{y_0 \sqrt{\tilde{\nu}_t}} e^{-\frac{1}{2y_0^2 \tilde{\nu}_t} \left( x_t - x_0 - \rho y_0 C_{RK}(\eta_t) \frac{\eta_t}{\nu} t^{\frac{1}{2}-H} \right)^2}$$

where $\tilde{\nu}_t = t \psi(\eta_t) := \{ (1 + \rho^2) C_{eR}(\eta_t) - \rho^2 C_{RK}^2(\eta_t) \} t$. 

Approximate distance function

Rewrite the joint density $p$ as

$$p(t, x_t, y_t | x_0, y_0) \approx \frac{1}{2\pi} \frac{1}{y_t \sqrt{\nu^2 t^2}} \frac{1}{y_0 \sqrt{\tilde{\nu}_t}} e^{-\frac{\tilde{d}^2(x_t, y_t | x_0, y_0)}{2t^{2H}}}$$

where

$$\tilde{d}(x_t, y_t | x_0, y_0) := \frac{\eta_t^2}{\nu^2} + \frac{1}{y_0^2 \psi(\eta_t)} \left( \frac{x_t - x_0}{t^{\frac{1}{2} - H}} - \rho y_0 C_{RK}(\eta_t) \frac{\eta_t}{\nu} \right)^2$$

is regarded as the approximate “distance function”.
Convexity of approximate distance function

Figure: The contour plots. Parameters $\rho = -0.7$, $\nu = 1$, $y_0 = 1$, $t = 0.5$. $H = 0.75$ on the right; $H = 0.25$, on the left.
Implied volatility approximation by bridge representation

By matching with the Black-Scholes price to the lowest order, we obtain a small time approximation of the implied volatility as follows. Let $\alpha = \frac{1}{2} - H$ and $k = \log \frac{K}{s_0}$.

Implied volatility approximation

$$\sigma_{BS}^2 \approx \frac{k^2}{T^{2\alpha}} \left\{ \frac{\eta^2}{\nu^2} + \frac{1}{y_0^2 \psi(\eta^*)} \left( \frac{k}{T^{\alpha}} - \rho y_0 C_{RK}(\eta^*) \frac{\eta^*}{\nu} \right)^2 \right\}^{-1}$$

where $\eta^*$ is the minimizer

$$\eta^* = \arg\min \left\{ \eta \in \mathbb{R} : \frac{\eta^2}{\nu^2} + \frac{1}{y_0^2 \psi(\eta)} \left( \frac{k}{T^{\alpha}} - \rho y_0 C_{RK}(\eta) \frac{\eta}{\nu} \right)^2 \right\}.$$ 

Note that $\eta^* = \eta^* \left( \frac{k}{T^{\alpha}} \right)$. 
Approximate implied volatility plots

Figure: The implied volatility curves. $t = 0.01$ on the left, $t = 1$ on the right. Parameters are set as $\rho = -0.4$, $\nu = 0.58$, $\alpha_0 = 0.38$. $H = 0.1$ in red, $H = 0.3$ in orange, $H = \frac{1}{2}$ in green, $H = 0.7$ in blue, $H = 0.9$ in purple.
Q: Does it recover the SABR formula to the lowest order when $H = \frac{1}{2}$?

**SABR formula**

\[
\sigma_{BS}(k) \approx -\nu k \frac{D(\zeta)}{D(\zeta)} , \quad \zeta = -\frac{\nu}{\alpha_0} k ,
\]

where

\[
D(\zeta) = \log \frac{\sqrt{1 - 2\rho \zeta + \zeta^2} + \zeta - \rho}{1 - \rho} .
\]
Q: Does it recover the SABR formula to the lowest order when $H = \frac{1}{2}$?

A: NO!

**SABR formula**

$$\sigma_{BS}(k) \approx \frac{-\nu k}{D(\zeta)}, \quad \zeta = -\frac{\nu}{\alpha_0} k,$$

where

$$D(\zeta) = \log \frac{\sqrt{1 - 2\rho \zeta + \zeta^2} + \zeta - \rho}{1 - \rho}.$$
Graphic comparison with SABR formula

**Figure**: The implied volatility curves from SABR and fSABR formula. Parameters are set as $\tau = 1$, $\rho = -0.06867$, $\nu = 0.58$, $\alpha_0 = 0.13927$. 
Recovery of SABR formula?

- Q: Does it recover the SABR formula to the lowest order when $H = \frac{1}{2}$?
  
  A: NO!

- Q: Maybe a smarter choice of $b_s$ might work?
Recovery of SABR formula?

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A: NO!

Q: Maybe a smarter choice of $b_s$ might work?
A: Unfortunately, doesn’t really work that way either.
Recovery of SABR formula?

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  A: NO!

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- Q: Is it even possible to recover the SABR formula from the bridge representation?
Recovery of SABR formula?

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  A: NO!

- Q: Maybe a smarter choice of $b_s$ might work?
  
  A: Unfortunately, doesn’t really work that way either.

- Q: Is it even possible to recover the SABR formula from the bridge representation?
  
  A: Most-likely-path from bridge representation
Bridge representation for multiperiod joint density

Notations

\[ t = (t_1, \cdots, t_n), \quad x_t = (x_{t_1}, \cdots, x_{t_n}), \quad y_t = (y_{t_1}, \cdots, y_{t_n}), \]
\[ B_t^H = (B_{t_1}^H, \cdots, B_{t_n}^H), \quad X_t = (X_{t_1}, \cdots, X_{t_n}), \quad Y_t = (Y_{t_1}, \cdots, Y_{t_n}), \]
\[ \xi_t = (\xi_{t_1}, \cdots, \xi_{t_n}), \quad \eta_t = (\eta_{t_1}, \cdots, \eta_{t_n}), \quad \zeta_t = (\zeta_{t_1}, \cdots, \zeta_{t_n}). \]

For \( 0 < t_1 < \cdots < t_n = T \), we are interested in obtaining a bridge expression for the multiperiod joint density

\[ p(x_{t_1}, y_{t_1}, \cdots, x_{t_n}, y_{t_n}) \]
\[ := \mathbb{P} [ (X_{t_1}, Y_{t_1}) = (x_{t_1}, y_{t_1}), \cdots, (X_{t_n}, Y_{t_n}) = (x_{t_n}, y_{t_n}) ]. \]
The multiperiod joint density $p$ has the following bridge representation

$$p(x_1, y_1, \ldots, x_n, y_n) = \mathbb{E} \prod_{k=1}^{n} e^{-\frac{1}{2y_0^2\bar{\rho}^2} \left( \Delta x_{t_k} - y_0 \rho \int_{t_{k-1}}^{t_k} e^{\nu B_t^H} dB_t + \frac{y_0^2}{2} \Delta \nu_{t_k} \right)^2} \left( 2\pi y_0^2 \bar{\rho}^2 \Delta \nu_{t_k} \right)^{-\frac{1}{2}} \left| \nu B_t^H = \eta_t \right| \times \mathbb{P} \left[ y_0 e^{\nu B_t^H} = y_t \right],$$

where $\Delta \nu_{t_k} = \nu_{t_k} - \nu_{t_{k-1}} = \int_{t_{k-1}}^{t_k} e^{2\nu B_s^H} ds$. 
Consider the log likelihood function

\[
\log p(x_{t_1}, y_{t_1}, \cdots, x_{t_n}, y_{t_n}) = \log \mathbb{E} \left[ \prod_{k=1}^{n} e^{\frac{-1}{2 y_0^2 \bar{\rho}^2 \Delta v_{t_k}} \left( \Delta x_{t_k} - y_0 \rho \int_{t_{k-1}}^{t_k} e^{\nu B_s^H} dB_s + \frac{y_0^2}{2} \Delta v_{t_k} \right)^2} \right]_{\nu B_t^H = \eta} + \log \mathbb{P} \left[ \nu B_t^H = \eta_t \right] - \sum_{k=1}^{n} \log y_{t_k}.
\]

- We calculate the first two terms in the limit as \( n \to \infty \) in the following.
Note that

$$\log \mathbb{P} \left[ \nu B^H_t = \eta_t \right] \approx -\frac{1}{2\nu^2} \eta' R^{-1} \eta,$$

where $R = [R(t_i, t_j)]$ is the covariance matrix of $B^H$. We approximate $R(t_i, t_j)$ by a Riemann sum

$$R(t_i, t_j) = \mathbb{E} \left[ B^H_{t_i} B^H_{t_j} \right] = \int_0^{t_i \wedge t_j} K(t_i, s)K(t_j, s)ds$$

$$\approx \sum_{k=0}^{i \wedge j} K(t_i, t_k)K(t_j, t_k)\Delta t = K'K\Delta t,$$

where $K$ is the upper triangular matrix

$$K_{ij} = \begin{cases} K(t_i, t_j), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$
Therefore, \( R^{-1} = K^{-1}(K')^{-1} \frac{1}{\Delta t} \).

Let \( b = (b_{t_1}, \ldots b_{t_n}) \) be the unique solution to the linear system

\[
\eta = Kb\Delta t \quad \iff \quad \eta_{t_i} = \nu \sum_{k=0}^{i} K(t_i, t_k) b_{t_k}\Delta t.
\]

Hence,

\[
\frac{1}{2\nu^2} \eta' R^{-1} \eta = \frac{1}{2} \Delta t b' K' R^{-1} K b \Delta t = \frac{1}{2} b' b \Delta t = \frac{1}{2} \sum_{k=1}^{n} b_{t_k}^2 \Delta t
\]

\[\rightarrow \frac{1}{2} \int_{0}^{T} b_{t}^2 \, dt \quad \text{as} \quad n \rightarrow \infty.\]

Also, in the limit \( n \rightarrow \infty \), \( \eta_t = \nu \int_{0}^{t} K(t, s) b_s \, ds \) for \( t \in [0, T] \).
As for the conditional expectation

$$
\log \mathbb{E} \left[ \prod_{k=1}^{n} e^{\frac{-1}{2y_0^2 \bar{\rho}^2 \Delta \nu t_k} \left( \Delta x_{t_k} - y_0 \rho \int_{t_k-1}^{t_k} e^{\nu B_s^H} dB_s + \frac{y_0^2}{2} \Delta \nu t_k \right)^2} \right] \bigg| \nu B_t^H = \eta
$$

$$
\approx \sum_{k=1}^{n} \mathbb{E} \left[ -\frac{1}{2y_0^2 \bar{\rho}^2 \Delta \nu t_k} \left( \Delta x_{t_k} - y_0 \rho \int_{t_k-1}^{t_k} e^{\nu B_s^H} dB_s \right)^2 \right] \bigg| \nu B_t^H = \eta
$$

Note that conditioned on $\nu B_t^H = \eta$, we have

$$
\Delta \nu t_k = \int_{t_{k-1}}^{t_k} e^{2\nu B_s^H} ds \approx e^{2\eta_{t_{k-1}}} \Delta t = e^{2\nu \sum_{j=0}^{k-1} K(t_{k-1}, t_j) b_{t_j} \Delta t} \Delta t
$$

and

$$
\Delta x_{t_k} - y_0 \rho \int_{t_{k-1}}^{t_k} e^{\nu B_s^H} dB_s \approx \Delta x_{t_k} - y_0 \rho e^{\eta_{t_{k-1}}} b_{t_{k-1}} \Delta t
$$

$$
= \left( \frac{\Delta x_{t_k}}{\Delta t} - y_0 \rho e^{\nu \sum_{j=0}^{k-1} K(t_{k-1}, t_j) b_{t_j} \Delta t} b_{t_{k-1}} \right) \Delta t
$$
Thus,

$$\sum_{k=1}^{n} \mathbb{E} \left[ -\frac{1}{2y_0^2 \rho^2} \Delta \nu_{t_k} \left( \Delta x_{t_k} - y_0 \rho \int_{t_{k-1}}^{t_k} e^{\nu B^H_s} dB_s \right)^2 \right] \left| \nu B^H = \eta \right.$$ 

$$\approx - \sum_{k=0}^{n} \frac{1}{2y_0^2 \rho^2 e^{2\nu} \sum_{j=0}^{k-1} K(t_{k-1},t_j) b_{t_j} \Delta t} \times$$

$$\left( \frac{\Delta x_{t_k}}{\Delta t} - y_0 \rho e^{\nu} \sum_{j=0}^{k-1} K(t_{k-1},t_j) b_{t_j} \Delta t b_{t_{k-1}} \right)^2 \Delta t$$

$$\rightarrow - \frac{1}{2} \int_{0}^{T} \frac{1}{y_0^2 \rho^2 e^{2\nu} \int_{0}^{t} K(t,s) b_s ds} \left( \dot{x}_t - y_0 \rho e^{\nu} \int_{0}^{t} K(t,s) b_s ds b_t \right)^2 dt$$

$$= - \frac{1}{2} \int_{0}^{T} \frac{1}{\rho^2 y_0^2 e^{2\eta_t}} \left( \dot{x}_t - \rho y_0 e^{\eta_t} b_t \right)^2 dt$$

as $n \rightarrow \infty$. 
Large deviations principle for fSABR

We end up with

\[ - \log \mathbb{P} [X_t = x_t, Y_t = y_t \text{ for } t \in [0, T]] = - \lim_{n \to \infty} \log p(x_{t_1}, y_{t_1}, \cdots, x_{t_n}, y_{t_n}) \]

\[ = \frac{1}{2} \int_0^T \frac{1}{\bar{\rho}^2 y_0^2 e^{2\eta_t}} (\dot{x}_t - \rho y_0 e^{\eta_t} b_t)^2 dt + \frac{1}{2} \int_0^T b_t^2 dt \]

\[ = \frac{1}{2} \int_0^T \frac{1}{\bar{\rho}^2 y_t^2} (\dot{x}_t - \rho y_t b_t)^2 dt + \frac{1}{2} \int_0^T b_t^2 dt \]

where \( b \in L^2[0, T] \) satisfying

\[ \eta_t = \log y_t - \log y_0 = \nu \int_0^t K_H(t, s) b_s ds \]

for \( t \in [0, T] \).
This should be the rate function for sample path LDP.
Recovery of Freidlin-Wentzell when $H = \frac{1}{2}$

Indeed,

$$b_t = \frac{1}{\nu} K^{-1}_H[\eta](t).$$

When $H = \frac{1}{2}$, $K^{-1}_H$ is simply the usual differential operator, thus

$$b_t = \frac{\dot{\eta}_t}{\nu} = \frac{1}{\nu} \dot{y}_t.$$ 

Therefore, the rate function reduces to

$$- \log \mathbb{P} [X_t = x_t, Y_t = y_t \text{ for } t \in [0, T]]$$

$$= \frac{1}{2} \int_0^T \frac{1}{\bar{\rho}^2 y_t^2} \left( \dot{x}_t - \rho y_t \frac{\dot{\eta}_t}{\nu} \right)^2 dt + \frac{1}{2} \int_0^T \left( \frac{\dot{\eta}_t}{\nu} \right)^2 dt$$

$$= \frac{1}{2} \int_0^T \frac{1}{\bar{\rho}^2 \nu^2 y_t^2} \left( \nu^2 \dot{x}_t^2 - 2 \rho \nu \dot{x}_t \dot{y}_t + \dot{y}_t^2 \right) dt$$

which recovers the classical large deviations principle of Freidlin-Wentzell.
Again, by matching with the Black-Scholes price, we obtain

\[
\sigma_{BS}^2 \approx \frac{k^2}{T} \left( \int_0^T \frac{1}{\rho^2 y_t^*^2} (\dot{x}_t^* - \rho y_t^* b_t^*)^2 + b_t^2 dt \right)^{-1},
\]

where \((x^*, b^*)\) is the minimizer

\[
(x^*, b^*) = \text{argmin} \left\{ \dot{x}, b \in L^2[0, T] : \int_0^T \frac{1}{\rho^2 y_t^2} (\dot{x}_t - \rho y_t b_t)^2 + b_t^2 dt \right\}
\]

with \(x_T = k\) and \(y_t^*\) is given by, for \(t \in [0, T]\),

\[
\log y_t^* - \log y_0 = \nu \int_0^t K_H(t, s) b_s^* ds
\]

This recovers the SABR formula when \(H = \frac{1}{2}\).
Conclusion

- We show a bridge representation for the (single and multi period) joint density of the lognormal SABR model.
- We obtained a full Edgeworth type of expansion in the uncorrelated case.
- Small time asymptotics to the lowest order are presented for both correlated and uncorrelated cases.
- A heuristic derivation of large deviations principle from multiperiod bridge representation for the fSABR model is shown, which recovers the classical Freidlin-Wentzell large deviations principle when $H = \frac{1}{2}$.
- Approximations of implied volatility in small time are obtained accordingly by matching terms with the Black-Scholes price.
- We emphasize that the fSABR formula also holds for general fSABR model, not only for lognormal fSABR.
THANK YOU FOR YOUR ATTENTION.