Time Series Analysis

4. Model free methods

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Outline

1. Time series in frequency domain
2. Singular spectrum analysis
3. Entropy methods

Time series in frequency domain

- So far, we have discussed various models within the*parametric approach* to time series analysis.
- The key element of this approach is to specify a time series model with a small (or moderate) number of free parameters which are determined via estimation from a data set.
- While this approach will remain the focus of these lectures, we will now take a brief side trip into the non-parametric (or model free) approach to time series analysis.
- In particular, we will focus of analyzing time series by means of expansion in various basis functions.
- The recurrent neural networks discussed at the end of this course fall into this category.

Time series in frequency domain

- The first approach that we discuss, namely *time series analysis in frequency domain* (in contrast to the *time domain* approach taken so far), is reminiscent of Fourier transform approach in signal processing.
- The idea is to decompose the underlying time series into components, each of which corresponds to evolution *cycles* of different frequencies.
- The appropriate basis functions are the trigonometric functions $\cos(\omega t)$ and $\sin(\omega t)$ or, equivalently, the complex exponential function $e^{i\omega t}$.

Spectral density function

- Let X_t be a covariance stationary time series, such that

$$\sum_{t=-\infty}^{\infty} |\Gamma_t| < \infty. \tag{1}$$

- The *spectral density function* (SDF), or *population spectrum*, of X_t is defined as

$$s_X(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} \Gamma_t e^{-i\omega t}. \tag{2}$$

  It is essentially the Fourier transform of Γ_t.

- From the trigonometric representation of complex numbers, and the fact that $\Gamma_{-t} = \Gamma_t$, we can write this in terms of purely real valued quantities:

$$s_X(\omega) = \frac{1}{2\pi} \Big(\Gamma_0 + 2 \sum_{t=1}^{\infty} \Gamma_t \cos(\omega t)\Big). \tag{3}$$

Spectral density function for white noise

- The easiest example is that of a white noise, $X_t = \varepsilon_t$. In this case,

$$\Gamma_t = \begin{cases} \sigma^2, & \text{if } t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- As a consequence, the SDF is constant,

$$s_X(\omega) = \frac{\sigma^2}{2\pi}\,. \tag{4}$$

Spectral density function for $AR(1)$

- As a next example, let us determine the spectral density function of the $AR(1)$ process. From equation (14) in Lecture Notes #1,

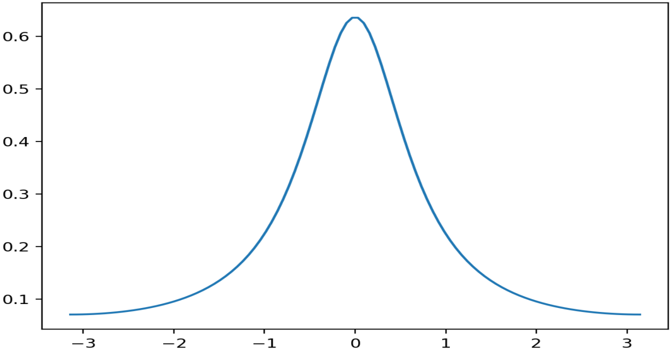
$$\begin{aligned} s_X(\omega) &= \frac{1}{2\pi}\sum_{t=-\infty}^{\infty}\Gamma_0\beta^{|t|}e^{-i\omega t} \\ &= \frac{\Gamma_0}{2\pi}\Big(1+\sum_{t=1}^{\infty}\beta^t e^{i\omega t}+\sum_{t=1}^{\infty}\beta^t e^{-i\omega t}\Big) \\ &= \frac{\Gamma_0}{2\pi}\Big(1+\frac{\beta e^{i\omega}}{1-\beta e^{i\omega}}+\frac{\beta e^{-i\omega}}{1-\beta e^{-i\omega}}\Big). \end{aligned}$$

- As a result,

$$s_X(\omega) = \frac{\sigma^2}{2\pi}\,\frac{1}{1-2\beta\cos\omega+\beta^2}\,. \tag{5}$$

Spectral density function for $AR(1)$

- Below is the plot of (5) with $\beta = 0.5$ and $\sigma = 1$.



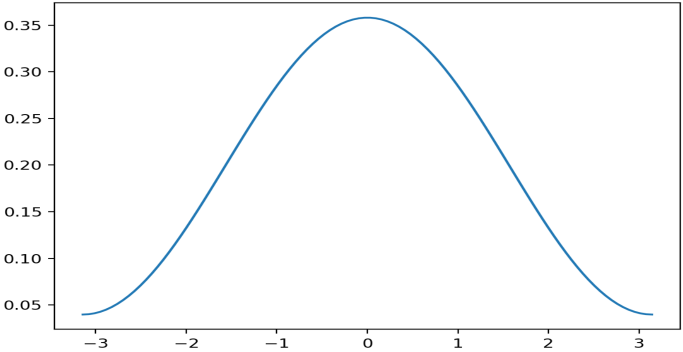
Spectral density function for *MA*(1)

- Let us now consider an *MA*(1) model. Using equation (62) in Lecture Notes #1, we see that
$$s_X(\omega)=\frac{1}{2\pi}\left((1+\theta^2)\sigma^2+\theta\sigma^2e^{i\omega}+\theta\sigma^2e^{-i\omega}\right).$$
- This implies that the SDF of an *MA*(1) process is

$$s_X(\omega)=\frac{\sigma^2}{2\pi}\left(1+2\theta\cos\omega+\theta^2\right). \tag{6}$$

Spectral density function for $MA(1)$

- Below is the plot of (6) with $\theta = 0.5$ and $\sigma = 1$.



Spectral density for *ARMA*(p, q)

- The calculations above can be generalized to produce an expression for the *ARMA*(p, q) model:
$$\psi(L)X_t = \alpha + \varphi(L)\varepsilon_t, \tag{7}$$
where our notation follows Lecture Notes #1.
- Namely, as you will show in Homework Assignment #5, the SDF is then given by
$$s_X(\omega) = \frac{\sigma^2}{2\pi}\left|\frac{\varphi(e^{i\omega})}{\psi(e^{i\omega})}\right|^2. \tag{8}$$
- If we factorize the polynomials $\psi(z)$ and $\varphi(z)$,
$$\begin{aligned}\psi(z) &= (1-\lambda_1 z)\ldots(1-\lambda_p z),\\ \varphi(z) &= (1-\mu_1 z)\ldots(1-\mu_q z),\end{aligned}$$
then
$$s_X(\omega) = \frac{\sigma^2}{2\pi}\,\frac{(1-2\mu_1\cos\omega+\mu_1^2)\ldots(1-2\mu_q\cos\omega+\mu_q^2)}{1-2\lambda_1\cos\omega+\lambda_1^2)\ldots(1-2\lambda_p\cos\omega+\lambda_p^2)}\,. \tag{9}$$

Spectral density function

- In general, the spectral density function $s_X(\omega)$ has the following properties:
  (i) It is non-negative.
  (ii) It is a periodic function of ω with period 2π (assuming $h = 1$).
  (iii) It is continuous in ω.
- The autocovariance can be calculated from the population spectrum by means of

$$\Gamma_t = \int_{-\pi}^{\pi} s_X(\omega) e^{i\omega t} d\omega. \tag{10}$$

- This is an immediate consequence of the fact that

$$\int_{-\pi}^{\pi} e^{i\omega(t-s)} d\omega = \begin{cases} 2\pi, & \text{if } t = s', \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

- Alternatively,

$$\Gamma_t = \int_{-\pi}^{\pi} s_X(\omega) \cos(\omega t) d\omega. \tag{12}$$

Spectral density function

- In particular,

$$\Gamma_0 = \int_{-\pi}^{\pi} s_X(\omega) d\omega, \tag{13}$$

  i.e. the variance of X_t is equal to the area under the population spectrum between $-\pi$ and π.
- This also leads to the interpretation of $s_X(\omega)$ as the fraction of the variance that is attributable to cycles of frequency ω.
- There is a general result that states that any covariance-stationary time series process can be expressed in terms of its spectral data.

Spectral representation theorem

- Namely, there exists a unique complex valued stochastic function $z_X(\omega)$, such that
$$X_t = \mu + \int_{-\pi}^{\pi} e^{i\omega t} z_X(\omega) d\omega, \tag{14}$$
where $\mu = \mathrm{E}(X_t)$.
- Since X_t is real valued, the random function $z_X(\omega)$ must have the following symmetry property:
$$\overline{z_X(\omega)} = z_X(-\omega). \tag{15}$$
- Furthermore, $z_X(\omega)$ has the following properties:
  (i) For all ω,
$$\mathrm{E}\big(z_X(\omega)\big) = 0. \tag{16}$$
  (ii) For all ω, ω',
$$\mathrm{E}\big(z_X(\omega)\overline{z_X(\omega')}\big) = s_X(\omega)\delta(\omega - \omega'), \tag{17}$$
  where $\delta(\omega - \omega')$ denotes Dirac's delta function.

Spectral representation theorem

- This result is known as the *spectral representation theorem* or *Cramer's theorem*.
- The spectral representation theorem can also be written in terms of real quantities only.
- Namely, we define

$$\begin{aligned} a_X(\omega) &= \mathrm{Re}\, z_X(\omega), \\ b_X(\omega) &= -\mathrm{Im}\, z_X(\omega) \end{aligned} \tag{18}$$

(the negative sign is just for convenience).

Spectral representation theorem

- Note that the random functions $a_X(\omega)$ and $b_X(\omega)$ have the following properties:

  (i)
$$\begin{aligned} a_X(-\omega) &= a_X(\omega), \\ b_X(-\omega) &= -b_X(\omega). \end{aligned} \tag{19}$$

  This is simply a conseqence of (15).

  (ii)
$$a_X(\omega)^2 + b_X(\omega)^2 = |z_X(\omega)|^2. \tag{20}$$

- As a result, we can write

$$X_t = \mu + \int_{-\pi}^{\pi} \left(cos(\omega t) a_X(\omega) + sin(\omega t) a_X(\omega) \right) d\omega. \tag{21}$$

Sample periodogram

- A complete proof of the spectral representation theorem is a bit technical, and can be found in specialized mathematical literature. Instead, we will interpret it in terms sample data.
- Let $x_1, \ldots, x_T$ be observations of X_t, and let $\widehat{\Gamma}_t$ denote the estimated autocovariance as defined by equation (5) in Lecture Notes #1. For any ω, the estimated sample spectral density function,

$$\widehat{s}_X(\omega) = \frac{1}{2\pi} \sum_{t=-(T-1)}^{T-1} \widehat{\Gamma}_t e^{-i\omega t}. \tag{22}$$

  is called the *sample periodogram*.
- We can then verify that

$$\widehat{\Gamma}_0 = \int_{-\pi}^{\pi} \widehat{s}_X(\omega) d\omega, \tag{23}$$

  i.e. the area under the periodogram is equal to the sample variance.

Sample periodogram

- In order to formulate the sample version of the spectral representation theorem, we assume that T is odd, and denote $\omega_j = 2\pi j/T$, for $j = -M, -M+1, \ldots, M$, where $M = (T-1)/2$.
- For each j, we define

$$\widehat{z}_X(\omega_j) = \frac{1}{T}\sum_{t=1}^{T} e^{-i\omega_j t} x_t - \widehat{\mu}. \tag{24}$$

Notice that

$$\widehat{z}_X(\omega_0) = 0. \tag{25}$$

- Then

$$x_t = \widehat{\mu} + \sum_{j=-M}^{M} e^{i\omega_j t}\,\widehat{z}_X(\omega_j). \tag{26}$$

Sample periodogram

- To see this, we multiply both sides of (24) by $e^{i\omega_j s}$ and sum over $j = 1, \ldots, M$, and notice that
$$\sum_{j=-M}^{M} e^{i\omega_j (s-t)} = \begin{cases} T, & \text{if } s = t, \\ 0, & \text{otherwise.} \end{cases}$$
- Finally, notice that
$$\sum_{j=1}^{T} (x_t - \widehat{\mu})^2 = \sum_{j=-M}^{M} |\widehat{z}_X(\omega_j)|^2. \tag{27}$$

Singular spectrum analysis

- *Singular spectrum analysis* (SSA) is a model free feature extraction methodology, which may be thought of as a variant of the principal component analysis (PCA).
- Its extension to multivariate time series (not discussed here) is referred to as *multi channel singular spectrum analysis* (M-SSA).
- We consider a sample from a time series $X_1, \ldots, X_T$, and let $1 < l < T$ be the length of the rolling window. Then $k = T - l + 1$ is the number of lagged vectors.
- The basic algorithm of SSA consists of two stages:
  (i) embedding,
  (ii) reconstruction.

Singular spectrum analysis

- Embedding is carried out in two steps. First, we form the *trajectory matrix*:

$$\mathcal{X} = \begin{pmatrix} X_1 & X_2 & \ldots & X_k \\ X_2 & X_3 & \ldots & X_{k+1} \\ \vdots & \vdots & \ldots & \vdots \\ X_l & X_{l+1} & \ldots & X_T \end{pmatrix}. \tag{28}$$

  Note that $\mathcal{X}_{ij} = X_{i+j-1}$; matrices of this form are called *Hankel matrices*.
- The columns in the trajectory matrix correspond to the observations of the time series as the length l observation window slides forward.

Singular spectrum analysis

- Then, we perform the singular value decomposition (SVD) of the trajectory matrix $\mathcal{X}$:
  (i) Let $\mathcal{S} = \mathcal{X}\mathcal{X}^{\mathrm{T}}$. Then $\mathcal{S}$ is positive definite; we denote its eigenvalues by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 0$, and the corresponding orthonormal system of eigenvectors by $U_1, U_2, \ldots, U_l$. The numbers $\sqrt{\lambda_i}$ are called the *singular values* of $\mathcal{X}$.
  (ii) Let $r = \mathrm{rank}(\mathcal{X})$ (typically, $r = l$), and set $V_i = \frac{1}{\sqrt{\lambda_i}}\mathcal{X}^{\mathrm{T}}U_i$, for $i = 1, \ldots, l$.
  (iii) Then
  $$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \ldots + \mathcal{X}_r, \tag{29}$$
  where $\mathcal{X}_i = \sqrt{\lambda_i}U_iV_i^{\mathrm{T}}$ are rank 1 matrices, called *elementary matrices*. The triple $(\sqrt{\lambda_i}, U_i, V_i)$ is called an *eigentriple* (ET) of the SVD and the vectors $\sqrt{\lambda_i}V_i = \mathcal{X}^{\mathrm{T}}U_i$ are the *principal components*.
  (iv) The numpy implementation of SVD is called `numpy.linalg.svd`.

Singular spectrum analysis

- The reconstruction stage is performed in two steps. First, we partition the set of indices $I = \{1, \ldots, r\}$ into m disjoint subsets $I = I_1 \cup \ldots \cup I_m$. For each subset I_k, form the sum

$$\mathcal{X}_{I_k} = \sum_{i \in I_k} \mathcal{X}_i. \tag{30}$$

  Clearly, this defines a decomposition of the trajectory matrix into components:

$$\mathcal{X} = \mathcal{X}_{I_1} + \ldots + \mathcal{X}_{I_m}. \tag{31}$$

- The final step is *diagonal averaging*. Each matrix $\mathcal{X}_{I_k}$ in the decomposition (31) is transformed into a new *reconstructed time series* $(\widetilde{X}_1^{(k)}, \widetilde{X}_2^{(k)}, \ldots, \widetilde{X}_T^{(k)})$ by means of the following procedure.

Singular spectrum analysis

- Let A be an $l \times k$-matrix, and let $T = l + k - 1$. We denote

$$A^*_{ij} = \begin{cases} A_{ij}, \text{ if } l < k, \\ A_{ji}, \text{ otherwise.} \end{cases} \tag{32}$$

Diagonal averaging transforms the matrix A into a time series $\widetilde{A}_1, \ldots, \widetilde{A}_T$ as follows:

$$\tilde{A}_j = \begin{cases} \frac{1}{j} \sum_{m=1}^{k} A^*_{m,j-m+1}, & \text{for } 1 \leq j < l \wedge k, \\ \frac{1}{l \wedge k} \sum_{m=1}^{l \wedge k} A^*_{m,j-m+1}, & \text{for } l \wedge k \leq j \leq l \vee k, \\ \frac{1}{N-j+1} \sum_{m=k-l \vee k+1}^{T-l \vee k+1} A^*_{m,j-m+1}, & \text{for } l \vee k \leq j \leq T. \end{cases} \tag{33}$$

- As a result, the original time series is represented as a sun of m reconstructed series;

$$X_t = \sum_{i=1}^{m} \widetilde{X}^{(i)}_t. \tag{34}$$

Singular spectrum analysis

- The choice of the rolling window length l is an important matter. It should be suffiently large so that each lagged time series incorporates the essential features of the original series $X_1, \ldots, X_N$.
- It is a good idea to perform SSA with different choices of l.

SSA of a simulated $I(1)$ process

- The figure below shows the results of SSA of the simulated $I(1)$ process given by the following specification:

$$X_t = 1.1 + X_{t-1} + 5.0\varepsilon_t, \tag{35}$$

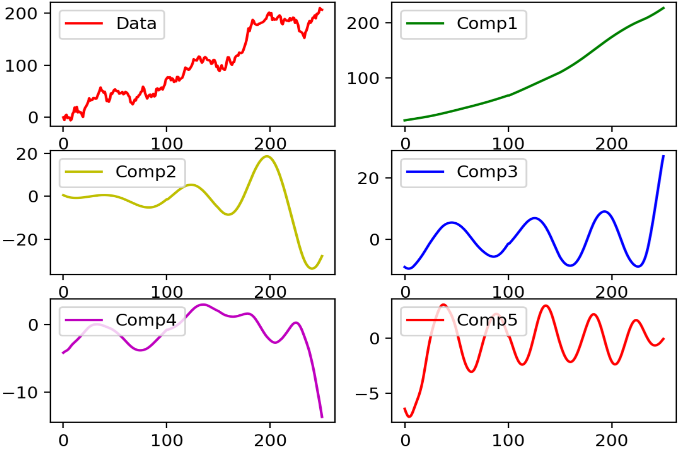
  where $\varepsilon_t \sim N(0,1)$.
- The upper left plot shows the actual time series, while the remaining ones show the first five SSA components.
- The cumulative weights, defined as

$$CW_j = \frac{\lambda_1 + \ldots + \lambda_j}{\lambda_1 + \ldots + \lambda_l}\,, \tag{36}$$

  of the plotted components are:

$$\begin{aligned} CW_1 &= 0.595,\\ CW_2 &= 0.653,\\ CW_3 &= 0.698,\\ CW_4 &= 0.720,\\ CW_5 &= 0.737. \end{aligned} \tag{37}$$

SSA of a simulated $AR(1)$ process



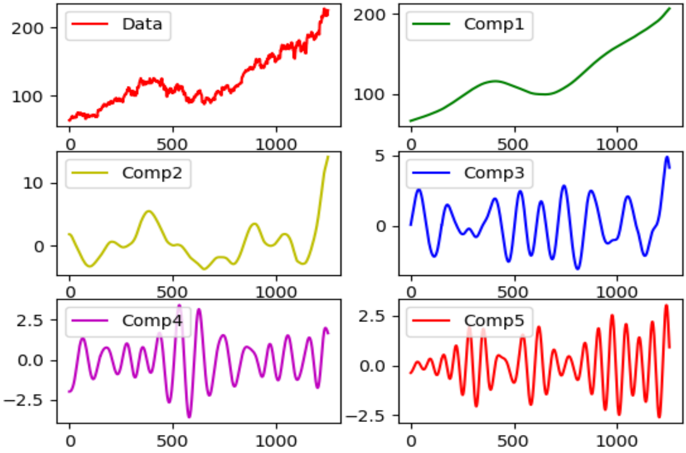
SSA of the AAPL share price

- The next figure below shows the results of SSA of the share price of Apple (AAPL) during the five-year period ending on September 28, 2018.
- As before, the upper left plot shows the actual time series, while the remaining ones show the first five SSA components.
- The weights of the plotted components are:

$$\begin{aligned} W_1 &= 0.769, \\ W_2 &= 0.033, \\ W_3 &= 0.016, \\ W_4 &= 0.013, \\ W_5 &= 0.011. \end{aligned} \tag{38}$$

- Notice that the first component (trend) is responsible for 76.9% of the dynamics.

SSA of the AAPL share price

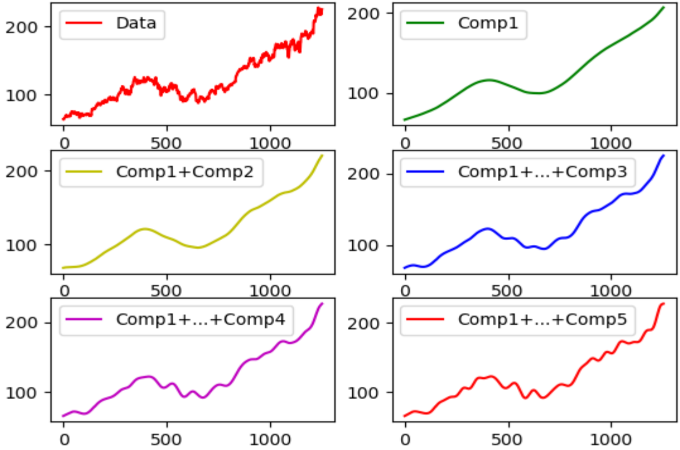


SSA of the AAPL share price

- Finally, the next figure shows the cumulative components of the dynamics of AAPL.
- The cumulative weights of the plotted components are:

$$\begin{aligned} CW_1 &= 0.769, \\ CW_2 &= 0.802, \\ CW_3 &= 0.818, \\ CW_4 &= 0.831, \\ CW_5 &= 0.852. \end{aligned} \tag{39}$$

SSA of the AAPL share price



Entropy

- The concept of Granger causality defined earlier in these lectures can be reformulated in terms of information transfer between two time series, using the concept of *transfer entropy*.
- Transfer entropy is defined in an essentially model free manner, lending itself to time series models beyond the *ARIMA* family.
- The price for the model freeness is a bit of formalism required. This formalism, the entropy methods, are extremely useful in data science, and we will first review them briefly.
- In order to lighten up on the math, we will sometimes be assuming that, for each t, X_t can take on only one of finitely many state values in $A = \{x_1, \ldots, x_K\}$.
- The probability of each of the states is denoted by p_i, $p_i = \mathrm{P}(X_t = x_i)$. Clearly,

$$\sum_{i=1}^{K} p_i = 1.$$

Entropy

- More generally, consider first a discrete random variable X, and let $p = (p_1, \ldots, p_K)$, $p_i = \mathrm{P}(X = x_i)$, denote its probability distribution.
- The *Shannon entropy* of the random variable X is defined by:

$$\mathrm{H}(X) = -\sum_{i=1}^{K} p_i \log p_i . \tag{40}$$

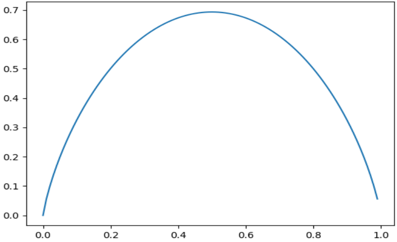
- The Shannon entropy has the following properties:
  (i) It is always nonnegative.
  (ii) Its value is 0, if one of the p_i's is 1.
  (iii) It reaches is maximum value $\log K$, if the distribution is uniform, $p_i = 1/K$, for all $i = 1, \ldots, K$.
- Shannon entropy is interpreted as a measure of information contained in the probability distribution: the lower the entropy, the higher its information content.

Entropy

- As an example, consider the case of a binary random variable, $K = 2$. Then $p = (w, 1 - w)$ and its entropy is given by the function:

$$h(w) = w\log(w) + (1 - w)\log(1 - w). \tag{41}$$

- Its graph is given below:



Entropy

- Is $X \in \mathbb{R}^n$ is a random variable with a continuous probability distribution $p(x)$, its Shannon entropy is defined by

$$\mathrm{H}(X) = -\int p(x) \log p(x) d^n x. \tag{42}$$

- For example, if $X \sim N(\mu, \sigma^2)$ is a normal random variable, then

$$\mathrm{H}(X) = \frac{1}{2} \log(2\pi e \sigma^2). \tag{43}$$

- In general, if $X \sim N(\mu, \Sigma)$ is a multivariate Gaussian random variable, then

$$\mathrm{H}(X) = \frac{1}{2} \log \det(2\pi e \Sigma). \tag{44}$$

Joint and conditional entropy

- Assume now that we have a joint (discrete) probability distribution $p_{i,j} = \mathrm{P}(X = x_i, Y = y_j)$, $j = 1, \ldots, K_1$, $j = 1, \ldots, K_2$, of two random variables X and Y.
- The *joint entropy* of X and Y is defined as

$$\mathrm{H}(X, Y) = -\sum_{i=1}^{K_1}\sum_{j=1}^{K_2} p_{i,j} \log p_{i,j}. \tag{45}$$

- Let $p_{i|j} = \mathrm{P}(X = x_i | Y = y_j)$ denote the conditional probability distribution of X given Y. The *conditional entropy* of X given Y is defined as

$$\mathrm{H}(X|Y) = -\sum_{i=1}^{K_1}\sum_{j=1}^{K_2} p_{i,j} \log p_{i|j}. \tag{46}$$

- The conditional entropy measures the information content in the probability distribution of X given the knowledge of Y.
- If X and Y are independent, then $\mathrm{H}(X|Y) = \mathrm{H}(X)$.

Joint and conditional entropy

- The joint and conditional entropies are related as follows:

$$\mathrm{H}(X, Y) = \mathrm{H}(Y) + \mathrm{H}(X|Y). \tag{47}$$

- *Proof:*

$$\begin{aligned}
\mathrm{H}(X, Y) &= -\sum_i \sum_j p_{i,j} \log p_{i,j} \\
&= -\sum_i \sum_j p_{i,j} \log p_{i|j} p_j \\
&= -\sum_i \sum_j p_{i,j} \log p_{i|j} - \sum_i \sum_j p_{i,j} \log p_j \\
&= -\sum_i \sum_j p_{i,j} \log p_{i|j} - \sum_j p_j \log p_j \\
&= \mathrm{H}(X|Y) + \mathrm{H}(Y).
\end{aligned}$$

Kullback-Leibler divergence

- Suppose now $q = (q_1, \ldots, q_K)$, $j = 1, \ldots, K$, is another probability distribution of the random variable X. This could possibly be an *a priori* guess of p or a parametric model of p.
- A measure of distance (or "divergence") between the distributions p and q, widely used in statistics and machine learning, is given by the *Kullback-Leibler divergence*, a.k.a. *relative entropy*:

$$\mathrm{KL}(p\|q) = \sum_{i=1}^{K} p_i \log \frac{p_i}{q_i}\,. \tag{48}$$

- For example, in the binary case, $p = (w, 1 - w)$, $q = (v, 1 - v)$,

$$\mathrm{KL}(p\|q) = w \log \frac{w}{v} + (1 - w) \log \frac{1 - w}{1 - v}\,.$$

Kullback-Leibler divergence

- For continuous probability distributions $p(x)$ and $q(x)$, their Kullback-Leibler divergence is defined by the integral

$$\mathrm{KL}(p\|q)=\int p(x)\log\frac{p(x)}{q(x)}\,dx\,. \tag{49}$$

- The Kullback-Leibler divergence gas the following properties:
  (i) $\mathrm{KL}(p\|q)\geq 0$.
  (ii) $\mathrm{KL}(p\|q)=0$ if and only if $p=q$.
- The proof is not hard, but it uses some properties of convex functions, which I will cover in detail in the Optimization Techniques in Finance course.
- Note that, unlike the conventional measure of distance, the Kullback-Leibler divergence is not symmetric in its arguments: $\mathrm{KL}(p\|q)\neq\mathrm{KL}(q\|p)$.

Mutual information

- The mutual information between two random variables is defined by

$$\mathrm{I}(X;Y)=\mathrm{H}(X)+\mathrm{H}(Y)-\mathrm{H}(X,Y). \tag{50}$$

- $\mathrm{I}(X;Y)$ has the following properties:
  (i)
  $$\mathrm{I}(X;Y)=\mathrm{I}(Y;X). \tag{51}$$
  (ii)
  $$\mathrm{I}(X;Y)=\mathrm{H}(X)-\mathrm{H}(X|Y). \tag{52}$$
  (iii)
  $$\mathrm{I}(X;Y)=\mathrm{H}(Y)-\mathrm{H}(Y|X). \tag{53}$$
  (iv)
  $$\mathrm{I}(X;X)=\mathrm{H}(X). \tag{54}$$

- *Proof:* Relations (52) and (53) are consequences of (47). The other relations are obvious.

Mutual information

- Mutual information measures the increase of information about X due to the available information about a random variable Y.
- If X and Y are independent, then $\mathrm{H}(X|Y)=\mathrm{H}(X)$ and $\mathrm{I}(X;Y)=0$ (nothing learned about X from Y).
- The mutual information of X and Y can explicitly be expressed in the form:

$$\mathrm{I}(X;Y)=\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}p_{i,j}\log\frac{p_{i,j}}{p_ip_j}\,, \tag{55}$$

which is the same as the Kullback-Leibler divergence between the joint distribution $p_{X,Y}$ and the product distribution p_Xp_Y,

$$\mathrm{I}(X;Y)=\mathrm{KL}(p_{X,Y},\,p_Xp_Y)\,. \tag{56}$$

Mutual information

- In other words, the mutual information of two random variables is a measure of distance between their joint distribution and the product of their respective marginals.
- In particular, mutual information is non-negative,

$$\mathrm{I}(X;Y)\geq 0. \tag{57}$$

- Finally, the *conditional mutual information* of X, Y given Z is defined by

$$\mathrm{I}(X,Y|Z)=\mathrm{H}(X|Z)-\mathrm{H}(X|Y,Z). \tag{58}$$

Transfer entropy

- Assume now that we are analyzing a (univariate or multivariate) time series model X_t.
- The concepts developed above can be applied to various random variables related to X_t, lagged values of X_t, etc.
- For example:
  - (i) The Shannon entropy of X_t is $\mathrm{H}(X_t)$.
  - (ii) The mutual information of two time series X_t and Y_t is $\mathrm{I}(X_t; Y_t)$.
  - (iii) An entropy measure capturing the dynamics of the time series over the period of j lags is given by $\mathrm{H}(X_t | X_{t-j:t-1})$.

Transfer entropy

- The *transfer entropy* from the process Y_t to X_t is defined as the mutual information of X_t and $Y_{t-j:t-1}$ conditioned on $X_{t-j:t-1}$:

$$\mathsf{T}(Y \to X) = \mathsf{I}(X_t, Y_{t-j:t-1} | X_{t-j:t-1}). \tag{59}$$

- In other words, transfer entropy from Y_t to Y_t measures the increase of information of X_t due to the inclusion of lagged information about Y_t, given lagged information about X_t.
- This also can be rewritten as

$$\mathsf{T}(Y \to X) = \mathsf{H}(X_t | X_{t-j:t-1}) - \mathsf{H}(X_t | X_{t-j:t-1}, Y_{t-j:t-1}). \tag{60}$$

- In the simplest case, if the two processes X_t and Y_t are independent, then, for any number of lags j,

$$p(x_t | x_{t-j:t-1}, y_{t-j:t-1}) = p(x_t | x_{t-j:t-1}), \tag{61}$$

and $\mathsf{T}(Y \to X) = 0$.

Transfer entropy

- In the case of a discrete valued process,

$$\mathrm{T}(Y \to X) = \sum_{x_{t-j:t}} \sum_{y_{t-j:t-1}} p(x_{t-j:t}, y_{t-j:t-1}) \log \frac{p(x_t | x_{t-j:t-1}, y_{t-j:t-1})}{p(x_t | x_{t-j:t-1})} . \quad (62)$$

- Transfer entropy is a very elegant and economic concept of causal dependence among time series.
- It applies to time series models that are not necessarily linear, or whose residuals are necessarily normally distributed.
- In case of autoregressive models with normally distributed disturbances, transfer entropy is essentially identical with the statistics used to test Granger causality.
- Namely, consider the bivariate, single lag model (39) of Lecture Notes #3.

Transfer entropy and Granger causality

- Transfer entropy with $j = 1$ is given by

$$\begin{aligned} \mathrm{T}(Y \to X) &= \mathrm{I}(X_t, Y_{t-1} | X_{t-1}) \\ &= \mathrm{H}(X_t | X_{t-1}) - \mathrm{H}(X_t | X_{t-1}, Y_{t-1}) \end{aligned} \tag{63}$$

- The terms on the right hand side can be evaluated by an explicit calculation. The result turns out to be

$$\mathrm{T}(Y \to X) = \frac{1}{2} \log \frac{\mathrm{Var}(\varepsilon_{1|f})}{\mathrm{Var}(\varepsilon_{1|p})} . \tag{64}$$

  Up to the constant $\frac{1}{2}$, this is the statistics used in the Granger causality test.

- The same concepts can be extended to multivariate time series, with the corresponding increase in the notational complexity.

Transfer entropy and Granger causality

- Estimation of transfer entropy from observed data is a bit of a challenge, as reliable estimates require large sample sets.
- Unlike the Granger test, which is a test on a linear regression coefficient, estimating transfer entropy requires information on the probability distributions of the processes.

References

[CT06] Cover, T. M., and Thomas, J. A.: *Elements of Information Theory*, Wiley (2006).

[GZ13] Golyandina, N., and Zhiglyavsky, A.: *Singular Spectrum Analysis for Time Series*, Springer (2013).

[H94] Hamilton, J. D.: *Time Series Analysis*, Princeton University Press (1994).

[S00] Schreiber, T.: Measuring Information transfer, *Phys. Rev. Lett.*, **85** ,461 - 464 (2000).