Optimization Techniques in Finance

5. Convex optimization. II

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Outline

1. Portfolio optimization problems
2. Numerical methods: unconstrained problems
3. Numerical methods: equality constrained problems
4. Numerical methods: inequality constrained problems

Mean variance optimization

- Our second group of examples of applications of convex optimization methods to financial problems is in the area of portfolio management.
- Consider a portfolio of risky assets $S_1, \ldots, S_n$, and let
  - (i) r_i denote the return on asset S_i,
  - (ii) $\mu_i = \mathrm{E}(r_i)$ denote the expected return on S_i,
  - (iii) $\sigma_i = \sqrt{\mathrm{Var}(r_i)}$ denote the standard deviation of returns on S_i,
  - (iv) ρ_{ij} denotes the correlation among the returns of S_i and S_j,
  - (v) $C \in \mathrm{Mat}_n(\mathbb{R})$ denotes the covariance matrix of returns on the assets, $C_{ij} = \rho_{ij}\sigma_i\sigma_j$.
  - (vi) w_i denotes the weight of asset S_i in the portfolio, $\sum_{i=1}^n w_i = 1$.

Mean variance optimization

- In the Markowitz *mean variance* portfolio problem, we are concerned with the question of allocating the assets in such a way, so that the variance of returns of the portfolio returns is minimal, while the expected return is at least a certain target level r.
- Additionally, one imposes other inequality and equality constraints which reflect the portfolio manager's (PM) mandate and views.
- For example, if the portfolio is long only, part of the inequality constraints will read $w_i \geq 0$.

Mean variance optimization

- In other words, the Markowitz *mean variance* optimization problem is formulated as the following convex optimization problem:

$$\min \ \sigma^2 = \frac{1}{2} w^{\mathrm{T}} C w \quad \text{subject to} \begin{cases} \mu^{\mathrm{T}} w \geq r, \\ Aw = b, \\ Bw \geq c. \end{cases} \tag{1}$$

- Let $\lambda_r, \lambda_{\mathcal{E}}, \lambda_{\mathcal{I}}$ denote the Lagrange multipliers corresponding to the three constraints above. The Lagrange function reads:

$$L(w, \lambda_r, \lambda_{\mathcal{E}}, \lambda_{\mathcal{I}}) = \frac{1}{2} w^{\mathrm{T}} C w - \lambda_r(\mu^{\mathrm{T}} w - r) - \lambda_{\mathcal{E}}^{\mathrm{T}}(Aw - b) - \lambda_{\mathcal{I}}^{\mathrm{T}}(Bw - c). \tag{2}$$

- Note that the signs of the Lagrange multipliers are consistent with our conventions explained in Lecture Notes #2.

Mean variance optimization

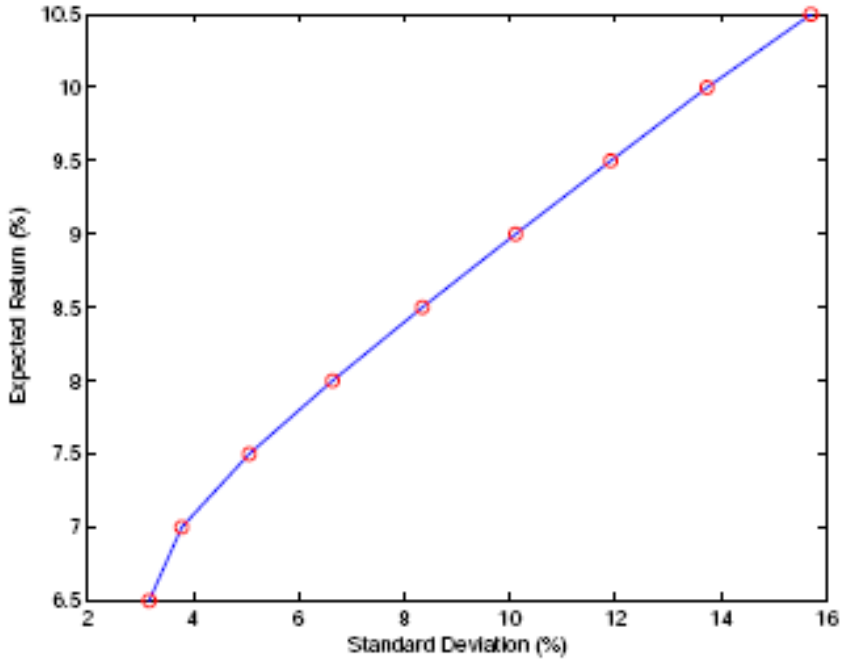
- The first order KKT conditions read:

$$\begin{aligned} Cw &= \lambda_r \mu + A^{\mathrm{T}} \lambda_{\mathcal{E}} + B^{\mathrm{T}} \lambda_{\mathcal{I}}, \\ \mu^{\mathrm{T}} w &\geq r, \\ Aw &= b, \\ Bw &\geq c, \\ \lambda_r(\mu^{\mathrm{T}} w - r) &= 0, \\ \lambda_{\mathcal{I}}(Bw - c) &= 0, \\ \lambda_r &\geq 0, \\ \lambda_{\mathcal{I}} &\geq 0. \end{aligned} \tag{3}$$

- Whether these conditions have a solution or not, and what is the optimal value, depends on the values of the parameters in the constraints.

Mean variance optimization

- In particular, there is only a certain range $r \in [r_{min}, r_{max}]$ of target expected portfolio returns, for which the problem has a solution.
- The curve representing the expected return r as a function of the optimal standard deviation σ of portfolio returns is called the *efficient frontier*.

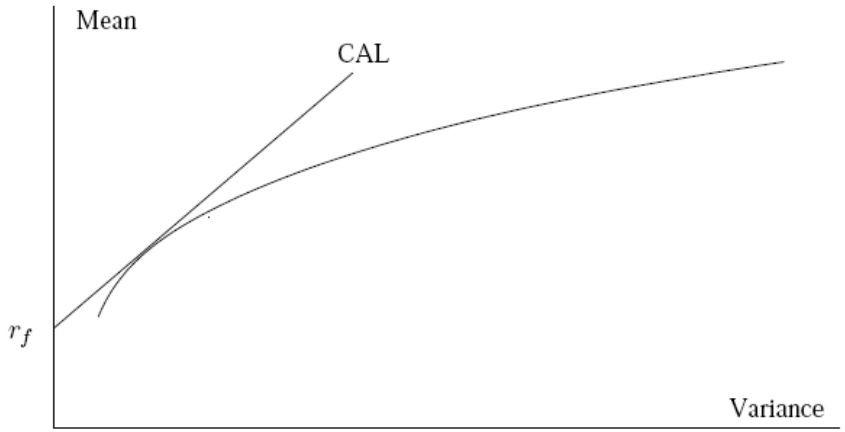


Mean variance optimization

- It is apparent that the solution of the optimality conditions (3) involves inverting the covariance matrix C.
- For this reason, it is very important that the estimated covariance matrix is well conditioned and has a well behaved inverse.
- As discussed earlier, a great deal of effort has been put into developing methodologies for covariance matrix estimation, especially for large portfolios.
- Without additional measures, (unconstrained) Markowitz MV optimization typically leads to highly unintuitive extreme portfolios, in which outsized long positions in some assets are offset by outsized short positions in other assets.
- Such portfolios, even though mathematically feasible, are practically totally unrealistic.
- Frequently (and IMHO incorrectly) this phenomenon is attributed to estimation errors of the "true" covariance matrix. These errors are supposed to be remediated by the regularization techniques discussed earlier.

Mean variance optimization

- Assume now that the portfolio contains a riskless asset with deterministic rate of return is r_f. It is natural to expect that $r_f < r$.
- We allocate a fraction $0 \leq \eta \leq 1$ of the portfolio into the riskless asset.
- Risk / return profiles for different values of η can be represented as a straight line, a *capital allocation line* (CAL), in the standard deviation / mean return plane.



- The optimal CAL lies below all the other CALs with $r > r_f$, since the corresponding portfolios have the lowest σ for any given value of $r > r_f$.

Mean variance optimization

- It follows that the optimal CAL goes through a point on the efficient frontier and never goes above a point on the efficient frontier.
- In other words, the slope of the optimal CAL is the derivative of the function $r(\sigma)$ that defines the efficient frontier, and so *the optimal CAL is tangent to the efficient frontier*.
- The point where the optimal CAL touches the efficient frontier corresponds to the optimal risky portfolio.

Maximizing the Sharpe ratio

- Alternatively, one can think of the optimal CAL as the one with the smallest slope. This is the portfolio that maximizes the *Sharpe ratio*:

$$\mathrm{SR}(w) = \frac{\mu^{\mathrm{T}} w - r_f}{\sqrt{w^{\mathrm{T}} C w}} \,. \tag{4}$$

- The corresponding optimization problem

$$\max \ \frac{\mu^{\mathrm{T}} w - r_f}{\sqrt{w^{\mathrm{T}} C w}} \quad \text{subject to} \ \begin{cases} Aw = b, \\ Bw \geq c \end{cases} \tag{5}$$

  is not concave, and it is hard to solve.
- It can be replaced with an equivalent convex quadratic problem as follows.

Maximizing the Sharpe ratio

- The feasible set of (5), $\mathcal{P} = \{w \in \mathbb{R}^n : \ Aw = b, Bw \geq c\}$, is a polyhedron. Define the convex set

$$\mathcal{P}^+ = \{(y, \kappa) \in \mathbb{R}^{n+1} : \ \kappa > 0, \text{ and } y/\kappa \in \mathcal{P}\} \cup (0, 0).$$

- Then the portfolio w^* optimizing (5) is of the form $w^* = y^*/\kappa^*$, where the pair (y^*, κ^*) is the solution to the following convex quadratic problem:

$$\min \ y^{\mathrm{T}} C y \quad \text{subject to} \begin{cases} (y, \kappa) \in \mathcal{P}^+, \\ (\mu - r_f)^{\mathrm{T}} y = 1. \end{cases} \tag{6}$$

- In order to see the equivalence of the two problems, we substitute in (6):

$$\kappa = \frac{1}{(\mu - r_f)^{\mathrm{T}} y},$$
$$y = \kappa w.$$

Benchmark tracking

- In quantitative asset management environments, portfolios are frequently selected with respect to a particular benchmark in mind.
- The benchmark may be a standard market index, such as S&P 500, or a customized index.
- The PM's mandate may be, for example, to closely track the benchmark, or outperform it by a certain amount.
- The *tracking error* (or *excess return*) of a portfolio with a given benchmark is the difference between the returns of the portfolio and the benchmark.
- It is defined as:

$$r^{\mathrm{T}} w - r^{\mathrm{T}} w_{bench} = r^{\mathrm{T}} (w - w_{bench}), \tag{7}$$

  where r^{T} is the vector of returns of the assets and w_{bench} is the vector of weights of the assets in the benchmark.
- The *ex ante* (predicted) tracking error is defined as

$$\epsilon(w) = \sqrt{(w - w_{bench})^{\mathrm{T}} C (w - w_{bench})} \tag{8}$$

Benchmark tracking

- The PM whose mandate is to track the benchmark with maximum ex ante tracking error ϵ faces the following convex optimization problem:

$$\max_{w} r^{\mathrm{T}}(w - w_{bench}) \quad \text{subject to} \quad \begin{cases} w^{\mathrm{T}} C w \leq \sigma^2, \\ (w - w_{bench})^{\mathrm{T}} C (w - w_{bench}) \leq \epsilon^2, \\ Aw = b, \\ Bw \geq c. \end{cases} \tag{9}$$

- Unlike the Markowitz problem (1) that has linear constraints only, it is not in standard quadratic programming (the constraint limiting the portfolio tracking error is quadratic), which makes it harder to solve.
- The tracking error constraint is, however, a convex quadratic function and, as discussed in Lecture Notes #4, we can rewrite this constraint in conic form.
- The resulting problem is a second-order cone optimization problem.

Benchmark tracking

- Since C is positive definite, there is a non-singular $R \in \mathrm{Mat}_n(\mathbb{R})$ such that $C = RR^{\mathrm{T}}$. We introduce the variables:

$$\begin{aligned} y_0 &= \sigma, \\ y &= R^{\mathrm{T}} w, \\ z_0 &= \epsilon, \\ z &= R^{\mathrm{T}}(w - w_{bench}). \end{aligned} \tag{10}$$

- With these definitions, the first two constraints in (9) say that the points $(y_0, y) \in \mathbb{R}^{n+1}$ and $(z_0, z) \in \mathbb{R}^{n+1}$ are elements of the second-order cone $\mathcal{K}_n$ in $\mathbb{R}^{n+1}$.

Benchmark tracking

- The benchmark tracking problem can be formulated as the following SOCP:

$$\max_{w} r^{\mathrm{T}}(w - w_{bench}) \quad \text{subject to} \begin{cases} Aw = b, \\ Bw \geq c, \\ R^{\mathrm{T}} w - y = 0 \\ R^{\mathrm{T}} w - z = R^{\mathrm{T}} w_{bench} \\ y_0 = \sigma, \\ z_0 = \epsilon, \\ (y_0, y) \in C_n, \\ (z_0, z) \in C_n. \end{cases} \tag{11}$$

Unconstrained convex problems

- We now move to discuss some of the algorithms used to solve convex optimization problems.
- We consider first an unconstrained convex optimization problem:

$$\min \ f(x), \tag{12}$$

where $f(x)$ is convex, twice continuously differentiable.

- As usual, by x^* we denote its solution, and by $f^* = f(x^*)$ its optimal value.
- From the analysis presented in Lecture Notes #4, the necessary and sufficient condition for x^* is

$$\nabla f(x^*) = 0. \tag{13}$$

Unconstrained strong convex problems

- As in the case of general nonlinear optimization problems, the solution methods are iterative, and start with an initial guess x_0 such that
  - (i) $x_0 \in \mathrm{dom}(f)$,
  - (ii) the *sublevel set* $S = \{x \in \mathrm{dom}(f) : \ f(x) \leq f(x_0)\}$ is closed.
- The second condition is usually hard to verify. Cases when it is true include:
  - (i) if $\mathrm{dom}(f) = \mathbb{R}^n$,
  - (ii) if $f(x) \to \infty$, as x approaches the boundary of $\mathrm{dom}(f)$.

Unconstrained strong convex problems

- A function $f(x)$ is *strongly convex* on C, if there exists $\mu > 0$ such that

$$\nabla^2 f(x) \geq \mu I, \quad \text{for all } x \in C. \tag{14}$$

- An implication of strong convexity is:

$$f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x) + \frac{1}{2}\mu \|y - x\|^2, \quad \text{for all } x, y \in C. \tag{15}$$

- Indeed, from Taylor's theorem, there is a z on the line segment connecting x and y such that

$$f(y) = f(x) + \nabla f(x)^{\mathrm{T}}(y - x) + \frac{1}{2}(y - x)^{\mathrm{T}} \nabla^2 f(z)(y - x),$$

which, together with (14), implies (15).

Unconstrained strong convex problems

- *Examples*:
  - (i) The function $f(x) = \frac{1}{2} x^{\mathrm{T}} P x$, where P is positive definite, is strongly convex. In this case, we can take μ to be the smallest eigenvalue of P.
  - (ii) The Kullback-Leibler divergence $f(p) = \mathrm{KL}(p\|q)$ is strongly convex with $\mu = 1$.
  - (iii) The function $f(x) = -\log(x)$ is convex but not strongly convex on $\mathbb{R}_+$.

Unconstrained strong convex problems

- We infer from (15) that

$$f^* \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2. \tag{16}$$

- To see this, we note that the RHS of (15) has a minimum in y for $\bar{y} = x - \nabla f(x)/\mu$, and the minimum value is equal to $f(x) - \|\nabla f(x)\|^2/2\mu$.
- This inequality allows us to formulate the following exit criterion for the search: in order to be within ε from f^*, $f^* - f(x) \leq \varepsilon$, we terminate the search when

$$\|\nabla f(x)\| \leq \sqrt{2\mu\varepsilon}.$$

Descent methods

- A *descent method* consists in constructing a sequence

$$x_{k+1} = x_k + t_k \Delta x_k, \tag{17}$$

where the search direction $\Delta x_k \in \mathbb{R}^n$ and step size $t_k > 0$ are chosen so that

$$f(x_{k+1}) < f(x_k). \tag{18}$$

- From convexity, Δx_k must satisfy

$$\nabla f(x_k)^{\mathrm{T}} \Delta x_k < 0, \tag{19}$$

i.e. $\nabla f(x_k)$ and Δx_k cannot be perpendicular to each other.

- General descent method: Choose an initial guess x_0 and iterate the following steps:

while(stopping criterion not satisfied)
  determine a descent direction Δx_k
  choose a step size t_k
  update $x_{k+1} = x_k + t_k \Delta x_k$

Backtracking line search

- The second step of the algorithm, the *line search*, determines where on the ray

$$\{x + t\Delta x : \ t > 0\} \tag{20}$$

the next iterate will be.

- We choose t to minimize the objective function along the ray (20):

$$t = \arg\min_{s>0} f(x + s\Delta x). \tag{21}$$

- It is usually sufficient to solve this problem approximately.
- A very simple and quite effective inexact line search method is the *backtracking line search*. It depends on two parameters $\alpha \in (0, 1/2)$ and $\beta \in (0, 1)$.
- Given a descent direction Δx and $t_0 = 1$, iterate the following steps:

$$\text{while } f\left(x + t_k \Delta x\right) \geq f(x) + \alpha t_k \nabla f(x)^{\mathrm{T}} \Delta x$$
$$t_{k+1} = \beta t_k$$

Backtracking line search

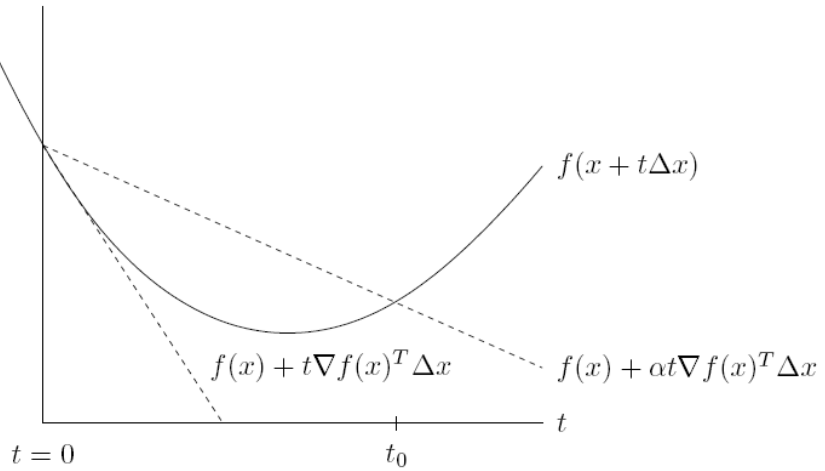
- The line search is called backtracking because it starts with unit step size and then reduces it by the factor β until the exit condition holds.
- Since Δx is a descent direction, we have $\nabla f(x)^{\mathrm{T}} \Delta x < 0$, and so

$$\begin{aligned} f(x + t\Delta x) &\approx f(x) + t\nabla f(x)^{\mathrm{T}} \Delta x \\ &< f(x) + \alpha t \nabla f(x)^{\mathrm{T}} \Delta x, \end{aligned}$$

  which shows that the backtracking line search eventually meets the stopping condition.
- The constant α can be interpreted as the fraction of the decrease in $f(x)$ predicted by linear extrapolation that we will accept.

Backtracking line search

- In the figure below, the lower dashed line shows the linear extrapolation of $f(x)$, and the upper dashed line has a slope a factor of α smaller.
- The backtracking condition is that $f(x)$ lies below the upper dashed line, i.e., $0 < t < t_0$.



Gradient descent methods

- In the gradient descent method we choose $\Delta x = -\nabla f(x)$.
- Choose an initial guess $x_0 \in \mathrm{dom}(f)$ and iterate:

  while(exit criterion not satisfied)
  $\Delta x_k = -\nabla f(x_k)$
  choose step size t_k via exact or backtracking line search
  update $x_{k+1} = x_k + t_k \Delta x_k$

- As already discussed in Lecture Notes #1, this method tends to be slow.

Steepest descent method

- The first order Taylor expansion of $f(x)$ is

$$f(x+\xi) \approx f(x) + \nabla f(x)^{\mathrm{T}} \xi.$$

- From calculus, $\nabla f(x)^{\mathrm{T}} \xi$ is the directional derivative of $f(x)$ in the direction of the vector ξ.
- We choose Δx to point in the direction of ξ, but we have to bound the magnitude of ξ.
- To this end, we choose a norm $\|x\|$ on $\mathbb{R}^n$ (for example, Euclidean), and define the *normalized steepest descent direction* as:

$$\Delta x_{\mathrm{nsd}} = \arg\min\{\nabla f(x)^{\mathrm{T}} \xi : \ \|\xi\| = 1\}. \tag{22}$$

- For example, if the norm $\|\xi\|$ is the Euclidean norm, then $\Delta x_{\mathrm{nsd}} = -\nabla f(x)$, and the method reduces to the gradient descent.

Steepest descent method: quadratic norm

- Given a positive definite matrix H, we define the quadratic H-weighted norm on $\mathbb{R}^n$ by

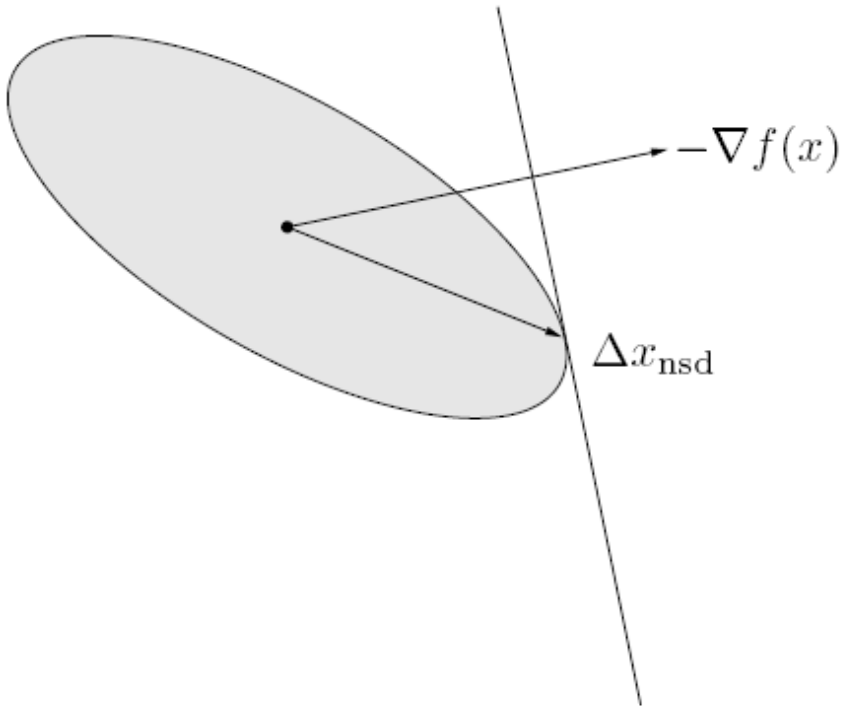
$$\|\xi\|_H = \sqrt{\xi^{\mathrm{T}} H \xi}. \tag{23}$$

- The normalized steepest descent direction is given by

$$\Delta x_{\mathrm{nsd}} = -H^{-1} \nabla f(x). \tag{24}$$

Steepest descent method: quadratic norm

- The ellipsoid shown in the figure below is the unit ball of the norm, translated to the point x. The normalized steepest descent direction Δx_{nsd} at x extends as far as possible in the direction $\nabla f(x)$ while staying in the ellipsoid.



Steepest descent method: L^1-norm

- We consider the steepest descent method for the L^1-norm. A normalized steepest direction,

$$\Delta x_{\text{nsd}} = \arg\min\{\nabla f(x)^{\mathrm{T}}\xi : \ \|\xi\|_1 = 1\}, \tag{25}$$

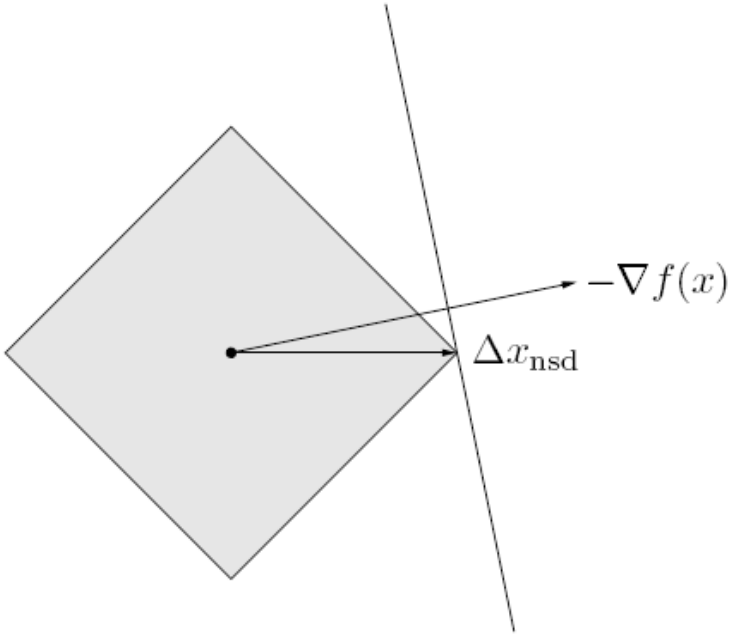
  can be characterized as follows.
- Let i be an index for which $\|\nabla f(x)\|_\infty = |\partial f(x)/\partial x_i|$. Then

$$\Delta x_{\text{nsd}} = -\text{sign}\left(\frac{\partial f(x)}{\partial x_i}\right) e_i, \tag{26}$$

  where e_i is the i-th standard basis vector.
- Thus, the normalized steepest descent step in L^1-norm can always be chosen to be a standard basis vector (or its negative).
- It is the coordinate axis direction along which the approximate decrease in $f(x)$ is greatest.

Steepest descent method: L^1 norm

- The diamond is the unit ball of the L^1-norm, translated to the point x. The normalized steepest descent direction can always be chosen in the direction of a standard basis vector. In the figure below, we have $\Delta x_{\text{nsd}} = e_1$.



- The steepest descent method in the L^1-norm has a natural interpretation. At each iteration we select a component of $\nabla f(x)$ with maximum absolute value, and decrease or increase the component x_i, according to the sign of $\big(\nabla f(x)\big)_i$.

Newton's methods

- For $x \in \mathrm{dom}(f)$, the step

$$\Delta x_{\mathrm{nt}} = -\nabla^2 f(x)^{-1} \nabla f(x) \tag{27}$$

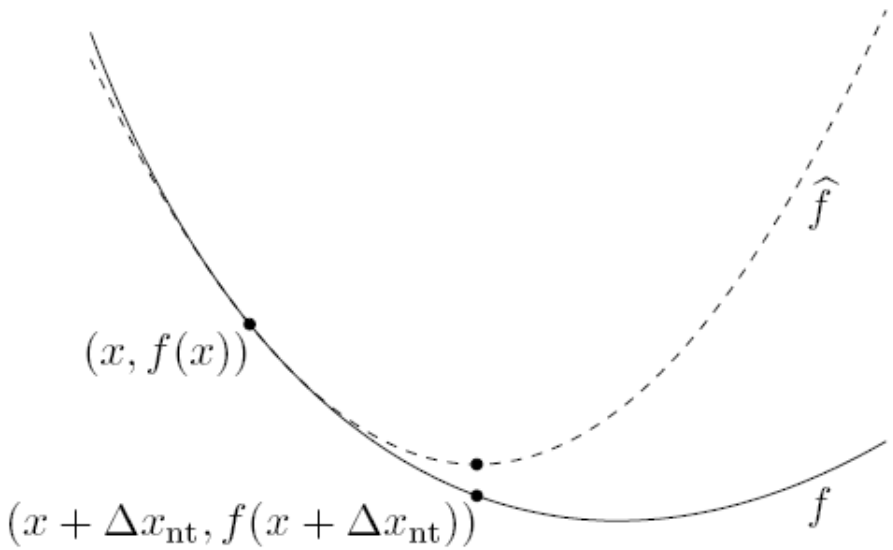
  is called the Newton step.
- The second order Taylor approximation to $f(x)$ is

$$\hat{f}(x+\xi) = f(x) + \nabla f(x)^{\mathrm{T}} \xi + \frac{1}{2} \xi^{\mathrm{T}} \nabla^2 f(x) \xi. \tag{28}$$

- This is a quadratic function in ξ, and $\xi = \Delta x_{\mathrm{nt}}$ is its minimizer!
- Since $f(x)$ is twice continuously differentiable, this *quadratic model* should be accurate for x near x^*.

Newton's method

- The figure below shows the function $f(x)$ (solid line) and its second order approximation $\hat{f}(x+\xi)$ (dashed). The Newton step Δx_{nt} is what must be added to x to give the minimizer of $\hat{f}(x+\xi)$.

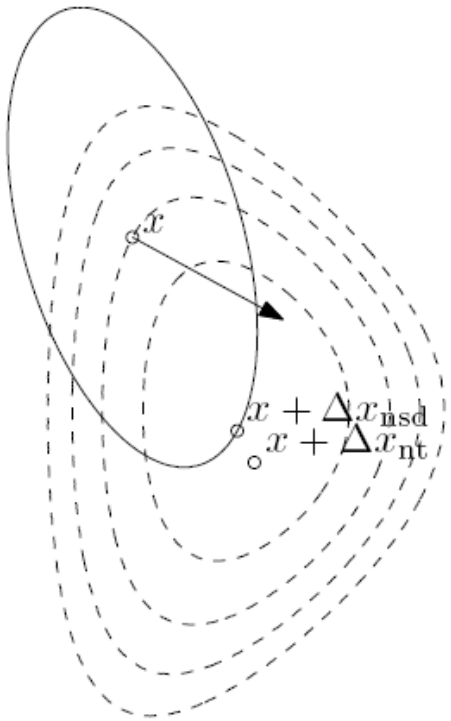


Newton's method

- We can get another insight into the workings of Newton's method by interpreting it in terms of the quadratic norm introduced in (23).
- Namely, the direction of the Newton step at x is the steepest descent direction for the norm (23) with $H = \nabla^2 f(x)$.

Newton's method

- The dashed lines are level curves of a convex function. The ellipsoid (solid line) is $\{x + \xi : \xi^T \nabla^2 f(x) \xi \leq 1\}$. The arrow shows $-\nabla f(x)$, the gradient descent direction. The Newton step Δx_{nt} is the steepest descent direction in the norm (23). The figure also shows Δx_{nsd}, the normalized steepest descent direction for the same norm.

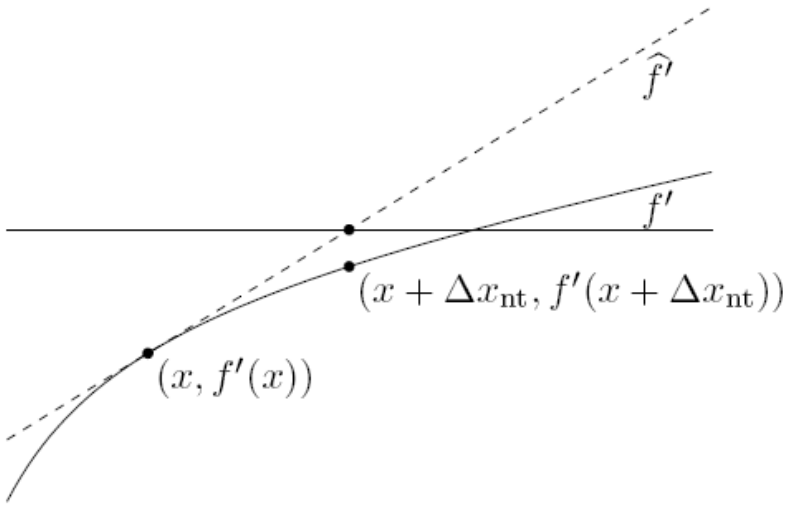


Newton's method

- The first order condition $\nabla f(x^*) = 0$ for x near x^* reads:

$$\nabla f(x) + \nabla^2 f(x)\xi \approx 0, \tag{29}$$

  and its solution is $\xi = \Delta x_{\text{nt}}$.
- Thus the Newton step is what must be added to x so that the linearized optimality condition holds. This is illustrated the figure below.
- The solid curve is the derivative $f'(x)$ of the function $f(x)$ shown in the previous figure, and $\hat{f}'(x)$ is the linear approximation of $f'(x)$. The Newton step Δx_{nt} is the difference between the root of $f'(x)$ and x.



Newton's method

- The quantity
$$\lambda(x) = \sqrt{\nabla f(x)^{\mathrm{T}} \nabla^2 f(x)^{-1} \nabla f(x)} \tag{30}$$
is called the *Newton decrement* at x.
- This is an important concept, because
$$\begin{aligned} f(x) - \min_{\xi} \hat{f}(x+\xi) &= f(x) - \hat{f}(x + \Delta x_{\mathrm{nt}}) \\ &= \frac{1}{2}\lambda(x)^2, \end{aligned}$$
and so it is a measure of distance between $f(x)$ and the minimum of its quadratic model.
- It can thus be used as an estimate of $f(x) - f^*$.

Newton's method

- We can summarize Newton's method in the following way.
- Choose an initial guess $x_0 \in \mathrm{dom}(f)$ and iterate the following steps:

while(stopping criterion not satisfied)
  compute the Newton step Δx_k and decrement $\lambda(x_k)$
  exit if $\lambda(x_k) < \varepsilon$
  choose a step size t_k by exact or backtracking line search
  update $x_{k+1} = x_k + t_k \Delta x_k$

Equality constrained problems: KKT conditions

- Consider now an equality constrained convex optimization problem:

$$\min f(x), \quad \text{subject to } Ax = b, \tag{31}$$

  where $f(x)$ is twice continuously differentiable.
- Without loss of generality we assume that $p = \operatorname{rank}(A) < n$ (so that the constraints are independent).
- If x^* is a solution to (31), the first oder conditions read:

$$\begin{aligned} \nabla f(x^*) + A^{\mathrm{T}} \lambda^* &= 0, \\ Ax^* &= b, \end{aligned} \tag{32}$$

  where λ^* is the vector of Lagrange multipliers.
- Solving (31) is equivalent to solving the system (32).

Equality constrained problems: KKT conditions

- For example, consider the quadratic problem:

$$\min \frac{1}{2} x^{\mathrm{T}} H x + q^{\mathrm{T}} x + r, \qquad \text{subject to } Ax = b, \tag{33}$$

  where H is positive semidefinite.
- The KKT conditions (32) read

$$\begin{pmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix}. \tag{34}$$

- This system may or may not have solutions depending on the problem. It is nonsingular, if the matrix $H + A^{\mathrm{T}} A$ is (strictly) positive definite.
- The system (34) is an example of a *KKT system*.

Newton's method

- Newton's method can be used to solve equity constraint convex problems! Key differences with unconstraint problems:
  - (i) the initial guess x_0 has to be feasible, i.e. $Ax_0 = b$,
  - (ii) the Newton step Δx_k has to satisfy the feasibility condition $A\Delta x_k = 0$.
- In order to find the appropriate step, we assume that x is feasible, and consider the second order Taylor approximation at x to the problem (31):

$$\min_{\xi} \; f(x) + \nabla f(x)^{\mathrm{T}}\xi + \frac{1}{2}\xi^{\mathrm{T}}\nabla^2 f(x)\xi, \qquad \text{subject to } A\xi = 0. \tag{35}$$

- This is a quadratic problem (in ξ) of the form (33), and its solution reduces to solving the linear system:

$$\begin{pmatrix} \nabla^2 f(x) & A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = \begin{pmatrix} -\nabla f(x) \\ 0 \end{pmatrix}. \tag{36}$$

Newton's method

- Newton's method for equality constraint convex problems can be formulated as follows.
- Choose an initial guess $x_0 \in \mathrm{dom}(f)$, such that $Ax_0 = b$ (i.e. x_0 is feasible) and iterate the following steps:

  while(exit criterion not satisfied)
  compute the Newton step Δx_k decrement $\lambda(x_k)$
  exit if $\lambda(x_k) < \varepsilon$
  choose the step size t_k by the backtracking line search

- With some effort, it is possible to extend this algorithm to infeasible starting points x_0. A detailed presentation can be found in [2].

Solving KKT systems

- We now describe methods for solving the (linear) KKT systems that arise in the process of determining the Newton step. We write this system in the general form

$$\begin{pmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = - \begin{pmatrix} q \\ p \end{pmatrix}. \tag{37}$$

- *Solving the full system.* This is the most straightforward approach. The KKT matrix is symmetric, but not necessarily positive definite, and so the preferred approach is the LDL^{T}-decomposition. This is a reasonable approach when the dimension of the problem is small.
- *Elimination.* Assuming that H is positive definite, we have

$$\begin{aligned} v &= -H^{-1}(q + A^{\mathrm{T}} w), \\ Av &= -p. \end{aligned}$$

and so

$$w = (AH^{-1}A^{\mathrm{T}})^{-1}(p - AH^{-1}q).$$

Solving KKT systems

- For convenience, we define the *Schur complement* of H:

$$S = -(AH^{-1}A^{\mathrm{T}})^{-1}. \tag{38}$$

- Notice that the matrix S is negative definite.
- A direct calculation shows that the solution above can be expressed in the block matrix form:

$$\begin{pmatrix} v \\ w \end{pmatrix} = - \begin{pmatrix} H^{-1} + H^{-1}A^{\mathrm{T}}SAH^{-1} & -H^{-1}A^{\mathrm{T}}S \\ -SAH^{-1} & S \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{39}$$

- The square block matrix on the right hand side is a special case of the *Schur inversion formula* of the block matrix on the left hand side of (40).
- In some cases (e.g. diagonal H), the Schur complement can be calculated efficiently, in which case this method is faster (linear in n rather than cubic) than the LDL^{T}-decomposition method.

Solving KKT systems

- *Elimination with singular H.* Problem 2 of Assignment #5 shows that the KKT matrix is nonsingular if and only if we can find some positive semidefinite matrix Q, such that $H + A^{\mathrm{T}} QA > 0$.
- Therefore, if $H \geq 0$ is singular, $H + A^{\mathrm{T}} QA > 0$, and the KKT matrix with H replaced by $H + A^{\mathrm{T}} QA > 0$ is nonsingular.
- But the system (40) is equivalent to the system:

$$\begin{pmatrix} H + A^{\mathrm{T}} QA & A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = - \begin{pmatrix} q + A^{\mathrm{T}} Qp \\ p \end{pmatrix}, \tag{40}$$

which is nonsingular and can be solved by elimination!

Formulation of the problem

- Consider now a general convex optimization problem

$$\min f(x), \quad \text{subject to } \begin{cases} c_i(x) \leq 0, \ i = 1, \ldots, m, \\ Ax = b. \end{cases} \tag{41}$$

  with $c_i(x)$ convex and twice continuously differentiable, and $p = \operatorname{rank}(A) < n$.

- We assume that
  - (i) the optimal solution x^* exists,
  - (iii) Slater's condition holds.

  As a result, λ^* exists and, along with x^*, satisfies the KKT conditions.

- The goal is to solve an inequality constrained problem by means of *interior point methods*, which reduce it to a sequence of linear constrained problems. We had a glimpse of this method in Lecture Notes #2.

Formulation of the problem

- We start by reformulating the problem as a logarithmic barrier problem:

$$\min f(x) + \frac{1}{t} B(x), \quad \text{subject to } Ax = b, \tag{42}$$

where $B(x)$ is the barrier function:

$$B(x) = -\sum_{i=1}^{m} \log\big(-c_i(x)\big) \tag{43}$$

- Note that this objective function is convex and twice continuously differentiable on its domain:

$$\begin{aligned} \nabla B(x) &= -\sum_{i=1}^{m} \frac{1}{c_i(x)} \nabla c_i(x), \\ \nabla^2 B(x) &= \sum_{i=1}^{m} \frac{1}{c_i(x)^2} \nabla c_i(x) \nabla c_i(x)^{\mathrm{T}} - \sum_{i=1}^{m} \frac{1}{c_i(x)} \nabla^2 c_i(x). \end{aligned} \tag{44}$$

Formulation of the problem

- As a result, Newton's method should be applicable!
- For example, in case of an inequality constrained LP problem, we are led to the following approximate optimization problem:

$$\min c^{\mathrm{T}} x - \frac{1}{t} \sum_{i=1}^{m} \log \left(b_i - a_i^{\mathrm{T}} x \right), \quad \text{subject to } Ax = b. \tag{45}$$

Choosing t

- One might expect that choosing large t right away might be a good idea. As we already mentioned in Lecture Notes #2, this is not necessarily the case.
- Newton's method works well if the Hessian of the objective function is not too large (Taylor's expansion!).
- From the explicit expression (44) we see, however, that the Hessian explodes as x approaches the boundary of the feasible set.
- For this reason, we will consider a sequence of problems with gradually increasing t, where each of the problems starts with the solution of the previous problem (warm start).

Central path

- For $t > 0$ we define the *central point* $x^*(t)$ as the solution to the optimization problem:

$$\min tf(x) + B(x), \quad \text{subject to } Ax = b, \tag{46}$$

  and assume that it exists.
- The *central path* is the set $\{x^*(t),\ t > 0\}$ of central points.
- The central path exists, provided the following conditions are satisfied for all $t > 0$:
  - (i) Strict feasibility:

$$\begin{aligned} c_i\left(x^*(t)\right) < 0, \ \text{ for } i = 1 \ldots, m, \\ Ax^*(t) = b. \end{aligned} \tag{47}$$

  - (ii) First order condition (*centrality condition*):

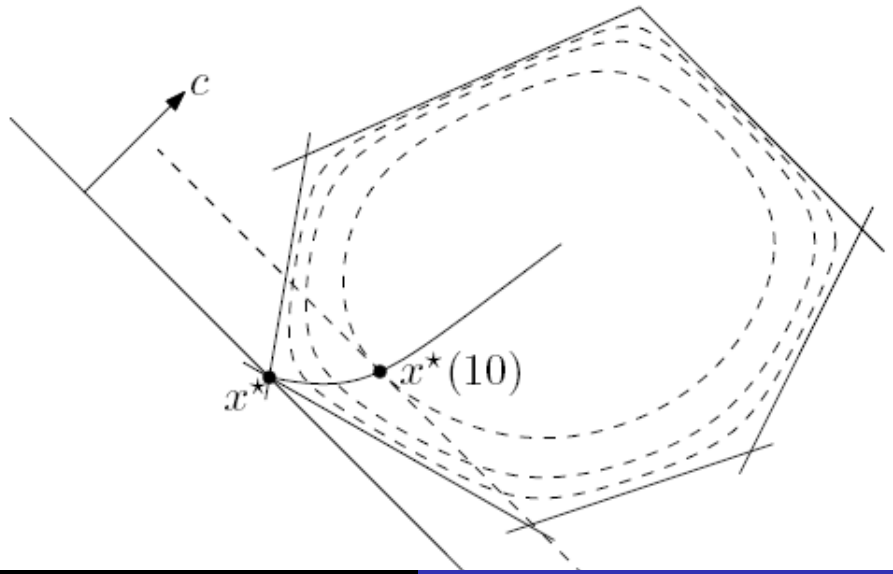
$$t\nabla f\left(x^*(t)\right) - \sum_{i=1}^{m} \frac{1}{c_i\left(x^*(t)\right)} \nabla c_i\left(x^*(t)\right) + A^{\mathrm{T}}\lambda = 0. \tag{48}$$

Central path

- In the LP example discussed above, the centrality condition reads:

$$tc + \sum_{i=1}^{m} \frac{1}{b_i - a_i^{\mathrm{T}} x} a_i + A^{\mathrm{T}} \lambda = 0. \tag{49}$$

- The figure below shows the central path for an LP with $n = 2$ and $m = 6$. The dashed curves show three contour lines of $B(x)$. The central path converges to x^* as $t \to \infty$. Also shown is the point on the central path with $t = 10$. The optimality condition (49) at this point can be verified geometrically. The line $c^{\mathrm{T}} x = c^{\mathrm{T}} x^*(10)$ is tangent to the contour line of $B(x)$ through $x^*(10)$.



Central path

- From the centrality condition (48) we can derive an important property of the central path: Every central point yields a dual feasible point, and hence a lower bound on the optimal value f^*.
- Specifically, define

$$\begin{aligned} \lambda_i^*(t) &= -\frac{1}{tc_i\big(x^*(t)\big)}\ , \ \text{for } i = 1, \ldots, m, \\ \nu_i^*(t) &= \frac{1}{t}\,\lambda_{m+i}, \ \text{for } i = 1, \ldots, p. \end{aligned} \tag{50}$$

- We claim that the pair $\lambda^*(t), \nu^*(t)$ is dual feasible, and so it yields a lower bound for the optimal value f^*.
- First of all, $\lambda_i^*(t) > 0$, since $c_i\big(x^*(t)\big) < 0$.

Central path

- Next, rewriting (48) as

$$\nabla f\big(x^*(t)\big) + \sum_{i=1}^{m} \lambda_i^*(t)\, \nabla c_i\big(x^*(t)\big) + A^{\mathrm{T}} \nu^*(t) = 0, \tag{51}$$

  we infer that $x^*(t)$ minimizes the Lagrange function:

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i c_i(x) + \nu^{\mathrm{T}}(Ax - b), \tag{52}$$

  for $\lambda_i = \lambda_i^*(t), \nu_i = \nu_i^*(t)$. This means that $\lambda^*(t), \nu^*(t)$ is indeed dual feasible.
- Let us now consider the dual Lagrange function $q\big(\lambda^*(t), \nu^*(t)\big)$:

$$\begin{aligned} q\big(\lambda^*(t), \nu^*(t)\big) &= f\big(x^*(t)\big) + \sum_{i=1}^{m} \lambda_i^*(t) c_i\big(x^*(t)\big) + \nu^*(t)^{\mathrm{T}}\big(Ax^*(t) - b\big) \\ &= f\big(x^*(t)\big) - \frac{m}{t}\,. \end{aligned}$$

Central path

- This identity says that the duality gap for $x^*(t)$ and $\left(\lambda^*(t), \nu^*(t)\right)$ is m/t.
- As $t \to \infty$, the duality gap goes to zero, i.e.

$$f\left(x^*(t)\right) - f^* \to 0. \tag{53}$$

- As a consequence, $x^*(t) \to x^*$, as $t \to \infty$.

The barrier method

- Algorithmically, the barrier method can be formulated as follows.
- Choose a strictly feasible initial guess $x_0 \in \mathrm{dom}(f)$, $t_0 > 0$, $\mu > 1$ and iterate the following steps:

  while(exit criterion not satisfied)
  compute $x^*(t_k)$ by minimizing $t_k f(x) + B(x)$ subject to $Ax = b$
  exit if $m/t_k < \varepsilon$
  define $t_{k+1} = \mu t_k$, and choose $x^*(t_k)$ to be the initial guess

- The first step of the algorithm is called the *centering step* or an *outer iteration*.
- The Newton iterations or steps executed during the centering step are referred to as *inner iterations*. At each inner step, we have a primal feasible point; we have a dual feasible point, however, only at the end of each outer (centering) step.

The barrier method

- We can estimate the number of outer iterations after which the algorithm will stop.
- Since it starts at t_0, and it exits when

$$\frac{m}{t_0 \mu^k} < \varepsilon \,,$$

the number of iterations is given by

$$k = \left\lceil \frac{\log\left(m/(\varepsilon t_0)\right)}{\log(\mu)} \right\rceil .$$

- While this analysis shows that the barrier method does converge (under reasonable assumptions), it does not address a basic question: As the parameter t increases, do the centering problems become more difficult?

The barrier method

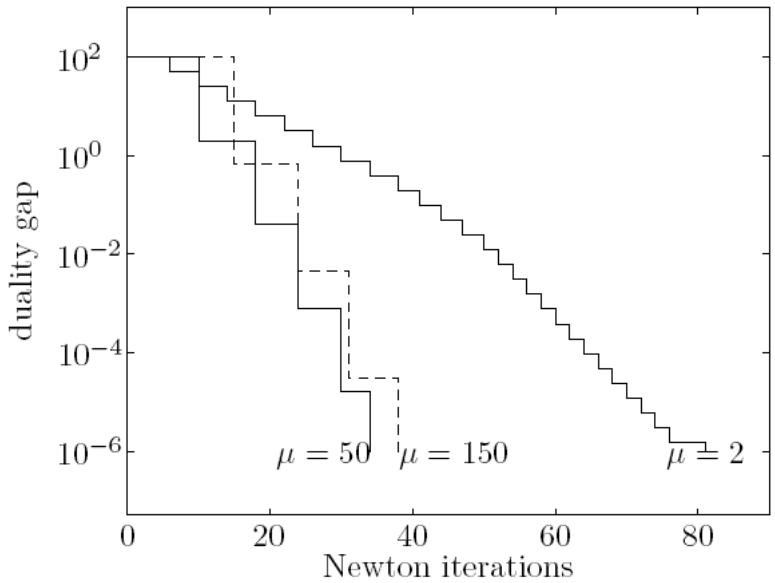
- Numerical evidence suggests that for a wide variety of problems, this is not the case; the centering problems appear to require a nearly constant number of Newton steps to solve, even as t increases.
- Computing $x^*(t)$ exactly is not necessary, since the central path has no significance beyond the fact that it leads to a solution of the original problem in the limit $t \to \infty$.
- On the other hand, the cost of computing an extremely accurate minimizer of $tf(x) + B(x)$, as compared to the cost of computing a good minimizer, is only marginally more, i.e., a few Newton steps at most. For this reason it is not unreasonable to assume exact centering.

The barrier method

- The choice of the parameter μ involves a trade-off in the number of inner and outer iterations required.
  - (i) If μ is close to 1 then, at each outer iteration, t_k increases by a small factor. As a result, the initial guess $x(t_k)$ for the next Newton search is a very good starting point, and the number of Newton steps needed to compute the next iterate is small. However, we expect a large number of outer iterations, since each outer iteration reduces the duality gap by only a small amount. In this case the iterates closely follow the central path.
  - (ii) If μ is large, after each outer iteration t_k increases by a large amount. Thus $x(t_k)$ may not be a very good guess for the next Newton search, and we expect many inner iterations. This results in fewer outer iterations, since the duality gap is reduced by the large factor μ at each outer iteration, but more inner iterations. With μ large, the iterates are widely separated on the central path.
- In practice, the two effects really offset each other. The total number of inner iterations are constant for sufficiently large μ. Values $10 \lesssim \mu \lesssim 20$ seem to work well.

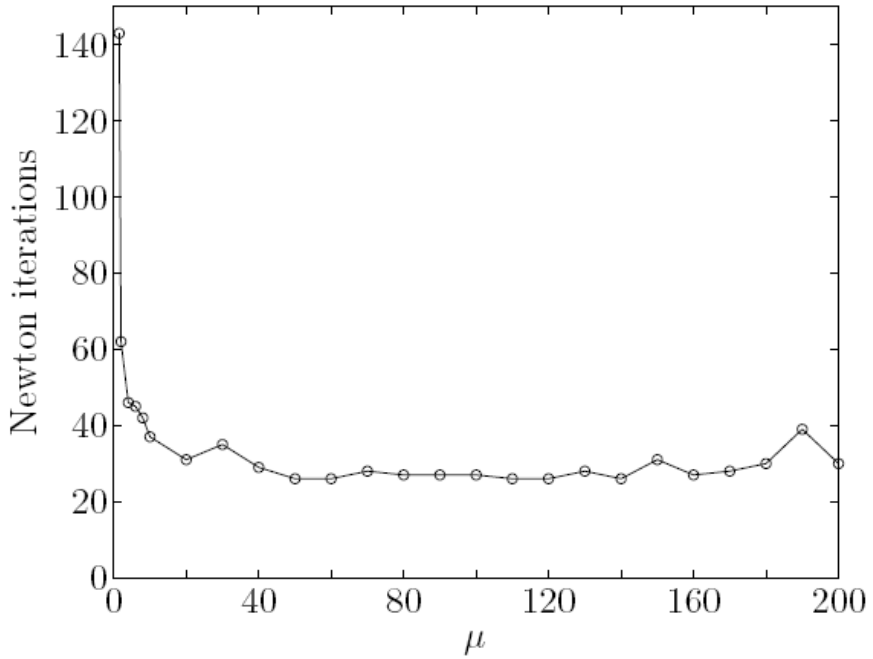
The barrier method

- The figure below shows the progress of barrier method for a small LP problem, showing duality gap versus cumulative number of Newton steps. Three plots are shown, corresponding to three values of the parameter μ: 2, 50, and 150. In each case, we have approximately linear convergence of duality gap.



The barrier method

- The figure below examines the trade-off in the choice of the parameter μ, for a small LP program. The vertical axis shows the total number of Newton steps required to reduce the duality gap from 100 to 10^{-3}, and the horizontal axis shows μ. The barrier method works well for values of μ larger than ≈ 3, but is otherwise not sensitive to the value of μ.



The barrier method

- An important issue is the choice of the initial value t_0 of t.
  - (i) If t_0 is chosen too large, the first outer iteration will require too many inner iterations.
  - (ii) If t_0 is chosen too small, the algorithm will require extra outer iterations, and possibly too many inner iterations in the first centering step.
- Since m/t_0 is the duality gap that results from the first centering step, it is reasonable to choose t_0 so that m/t_0 is approximately of the same order as $f\big(x^*(0)\big) - f^*$, or μ times this amount.
- For example, if a dual feasible point (λ, ν) is known, with duality gap $\eta = f\big(x^*(0)\big) - q(\lambda, \nu)$, then we can take $t_0 = m/\eta$. Thus, in the first outer iteration we simply compute a pair with the same duality gap as the initial primal and dual feasible points.

Final remarks

- The methods discussed in this section (central paths, barrier method, etc) can be extended to include conic optimization problems such as as SOCP and SDP discussed in Lecture Notes #4 and #5.
- *Primal-dual interior point methods* are often more efficient than the barrier method, especially when high accuracy is required.
  (i) They update primal and dual variables at each iteration; there is no distinction between inner and outer iterations.
  (ii) They often exhibit superlinear asymptotic convergence.
  (iii) Iterations can start at infeasible points.
  (iv) The cost per iteration same as for the barrier method.
- These topics are discussed in [2].

Recap

- Surprisingly many problems in finance can be expressed as convex optimization problems.
- Roughly speaking, tractability in optimization requires convexity: local optima are global.
- Unlike convex problems, algorithms for nonconvex optimization find local (often suboptimal) solutions (Levenberg-Marquardt, BFGS, ...), or are very expensive (differential evolution, ...).
- Interior-point methods require a small number of steps (20 – 80 steps in practice).
- Basic algorithms (Newton, barrier, . . .) are easy to implement and work well for small and medium size problems (and larger problems if the structure is exploited).

Credits

- The theoretical and algorithmic parts of Lecture Notes #4 – #5 follow to a large degree the presentation of [2].
- All figures have been copied from that book.

References

[1] Bertsekas, D. P.: *Nonlinear Programming*, Athena Scientific (2016).

[2] Boyd, S., and Vanderberghe, L.: *Convex Optimization*, Cambridge University Press (2004).

[3] Cornuejos, G., and Tutuncu, R.: *Optimization Methods in Finance*, Cambridge University Press (2007).