Market Impact with Autocorrelated Order Flow Under Perfect Competition: The Donier Model

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Market Microstructure: Confronting Many Viewpoints,
December 11, 2012
By \textit{metaorder}, we mean an order that is sufficiently large that it cannot be filled immediately without eating into the order book.

- Nowadays, this means just about any order.
- Such orders need to be split.

We refer to each component of a metaorder as a \textit{child order}.

By the \textit{metaorder impact profile} (or just \textit{impact profile}), we mean the average path of the stock price during and after execution of a metaorder.

\textit{Completion} refers to the timestamp of the last child order of a metaorder.
Schematic of the metaorder impact profile

Figure 1: The metaorder impact profile
The impact profile

- When a buy metaorder is sent, its immediate effect is to move the price upwards (to $p$ say).
- After completion, the price reverts to some price $p_\infty$.
- Market impact then has two components, one transient and one permanent.
- Knowledge of the metaorder impact profile is key to the derivation of optimal execution strategies.

- The Donier model provides a framework for understanding and quantifying the impact profile.
Price impact of metaorders

It is now (fairly) well-accepted that:

- Metaorder price impact is a power-law function of the quantity, close to square-root.
- Prior to completion, the impact profile is a power-law function of time, close to square-root.
- After completion, market impact decays, possibly to a permanent level.
Impact of proprietary metaorders (from Tóth et al.)

Figure 2: Log-log plot of the volatility-adjusted price impact vs the ratio $Q/V$
Notes on Figure 2

- In Figure 2 which is taken from [Tóth, Bouchaud et al.], we see the impact of metaorders for CFM\(^1\) proprietary trades on futures markets, in the period June 2007 to December 2010.
  - Impact is measured as the average execution shortfall of a metaorder of size \(Q\).
  - The sample studied contained nearly 500,000 trades.

- We see that the market impact is empirically power-law (roughly square-root) for metaorders with a range of sizes spanning two to three orders of magnitude!

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\(^1\) Capital Fund Management (CFM) is a well-known large Paris-based hedge fund.
Empirical impact profile (from Moro et al.)

Figure 3: Average path of the stock price during execution of a metaorder on two exchanges
From Figure 3, we see that

- There is reversion of the stock price after completion of the metaorder.
- Some component of the market impact of the metaorder appears to be permanent.
- The path of the price prior to completion looks like a power law.
  - From [Moro et al.]

\[
m_t - m_0 \approx (4.28 \pm 0.21) \left( \frac{t}{T} \right)^{0.71\pm0.03} \quad \text{(BME)}
\]

\[
m_t - m_0 \approx (2.13 \pm 0.05) \left( \frac{t}{T} \right)^{0.62\pm0.02} \quad \text{(LSE)}
\]

where \( T \) is the duration of the metaorder.
Summary of empirical observations

- The square-root formula gives an amazingly accurate rough estimate of the cost of executing an order.
- During execution of a metaorder, the price moves on average roughly according to \((t/T)^{2/3}\).
- Immediately after completion of a metaorder, the price begins to revert.
- The impact profile seems to exhibit scale invariance (which is implicit in Figure 3).
Let $\epsilon_t$ be the sign of the child order observed at time $t$.

Then the autocorrelation function is given by $\rho(\tau) = \langle \epsilon_t \epsilon_{t+\tau} \rangle$.

By assumption, if two child orders come from different metaorders, their order signs are uncorrelated.

$p(L)$ is the probability that a metaorder has length $L$.

We assume there are always $N$ active metaorders.
The power-law case

In the realistic case where metaorder sizes $L$ are power-law distributed so

$$p(L) \sim L^{-(1+\gamma)}$$

we find

$$\rho(\tau) \sim N\gamma^{-2} \tau^{1-\gamma}.$$ 

In particular, if $\gamma = 3/2$ as is more or less the case empirically for many stocks, we have

$$\rho(\tau) \sim \frac{1}{\sqrt{\tau}}.$$ 

- The LMF model gives a link between the distribution of order sizes and the autocorrelation function of order signs.
Empirical confirmation

- [Tóth, Lillo et al.] perform a careful analysis of order flow data from the London Stock Exchange containing exchange membership identifiers.
- They conclude that order splitting is indeed the dominant cause of the long memory of the order sign process.
The Glosten and Milgrom sequential trade model

In the [Glosten and Milgrom] model, the market maker $\mathcal{M}$ learns the informed trader $\mathcal{I}$’s information by observing the order flow.

- If there are more buys than sells over time, $\mathcal{M}$ sets the price higher.

Under perfect competition:

- $\mathcal{M}$ posts bid and ask prices $B = \mathbb{E}[V | \text{Sell}]$ and $A = \mathbb{E}[V | \text{Buy}]$ where $V$ denotes the efficient price.
- The spread $s = A - B$ is proportional to the probability $\mu$ of informed trading.
Dynamical properties of the Glosten and Milgrom model

The trade price series is a martingale.

- Both bid and ask prices are expectations conditioned on an expanding information set (the time series of trade signs):

$$B_k = \mathbb{E}[V | F_k, \epsilon_k = -1]$$
$$A_k = \mathbb{E}[V | F_k, \epsilon_k = +1]$$

Order signs are predictable: $\mathbb{E}[\epsilon_k | F_k] \neq 0$ in general.

Orders are serially correlated because informed traders always trade in the same direction.

There is market impact in this model. A buy causes both the bid and the offer to increase.
The FGLW market impact model

In the model of [Farmer, Gerig, Lillo, and Waelbroeck] (FGLW from now on), there is a market maker $\mathcal{M}$, informed traders $\mathcal{I}$ and uninformed (or noise) traders $\mathcal{U}$. Informed traders trade using metaorders.

The authors show that the typical impact profile associated with the execution of a metaorder may be recovered starting from two assumptions:

- **The Martingale Condition**: The price process is a martingale.
  - $\mathcal{M}$ does not know how long a given metaorder will continue.

- **The Fair Pricing Condition**: On average, the price reverts after completion of the metaorder to a level equal to the average price paid by $\mathcal{I}$.
  - If metaorder sizes are power-law distributed with exponent $\gamma$, the price reverts on average to a level which is a factor $1/\gamma$ of the peak price reached at completion.
In a (very!) recent paper, [Bershova and Rakhlin] perform an empirical study of a proprietary dataset of large institutional equity orders.

They broadly confirm the predictions of FGLW, including the power-law impact profile, the reversion level of 2/3 and roughly square-root permanent impact.

They also study reversion of market impact in detail.
The CFM model of latent supply and demand

- [Tóth, Bouchaud et al.] introduce the concept of *latent* supply and demand, to be distinguished from the visible supply/demand profile associated with the limit order book.

- [Tóth, Bouchaud et al.] note that if latent supply and demand is roughly linear in price over some reasonable range of prices, market impact should be roughly square-root.
The Donier model

In the model of [Donier]:

- As before, the market consists of informed traders $\mathcal{I}$, noise traders $\mathcal{U}$, and market makers $\mathcal{M}$.
- There is *perfect competition* between market makers. We may thus suppose there is only one market maker $\mathcal{M}$.
- Informed traders submit metaorders generating autocorrelation in the order sign process according to the order-splitting model of [Lillo, Mike and Farmer].
- $\mathcal{M}$ has *perfect information* and can distinguish informed and uniformed child orders as in the colored print model of [Gerig, Farmer and Lillo].
  - As before, $\mathcal{M}$ does not know in advance when $\mathcal{I}$'s metaorder will complete.
  - $\mathcal{M}$ does however know the distribution of metaorder sizes.
Market making strategy

The market maker reacts to a child order as follows:

- If the child order is uniformed, $M$ responds by refilling the order book so as to restore it to its original state.
- If the child order is informed, $M$ lets the child order eat into the order book.
  - The zero profit condition imposes the quantity $v_p$ that should be available at the price level $p$.
- Only informed trades can cause the price to move.
Computation of $v_p$

- Let $\ell$ denote the (unknown) length of the metaorder.
- $\mathcal{M}$ posts limit orders such that the quantity $v_p = L_p - L_{p-1}$ available at price $p$ satisfies

$$p = \mathbb{E}[p_\infty | \ell \geq L_p]$$

where $p_\infty = \lim_{t \to \infty} p_t$.

- Note that the $v_p$ limit orders could be posted either in advance or at the time in response to incoming market orders. The $v_p$ thus represents latent limit order supply.
- Strictly speaking, our argument works only if informed orders have length $L_k$ for some $k$. We believe that a more careful analysis will not substantially change the results.
We have assumed perfect information for convenience; only informed trades move the price which is thus a deterministic function of $L$.

- The impact profile and in particular the final price $p$ and the reversion price $p_\infty$ are deterministic.

In reality, $M$ cannot tell whether a given child order is informed or not. Information is asymmetric and noise trades also move the price.

- The deterministic impact profile, $p$ and $p_\infty$ should then be viewed as expectations of actual prices.

$\mathcal{I}$ is informed only in the sense that he knows what the size of his own metaorder is. This is of course $\mathcal{I}$’s private information.
Let $\ell$ be the \textit{(a priori unknown)} length of the metaorder and $p_{\text{max}}$ the price reached at completion.

Then, either

- $\ell > L + \nu_p$ and the metaorder exhausts the available quantity at $p$, causing the price to increase to $p + 1$, or
- $L \leq \ell \leq L + \nu_p$, $p = p_{\text{max}}$, the metaorder completes, and the price reverts to $p_\infty$.

Denote the probability that the metaorder continues by $q$.

Then because $p_t$ is a martingale,

$$p = (1 - q) p_\infty + q (p + 1) \quad \text{(2)}$$

and the decay after completion (\textit{transient impact}) is

$$\Delta(p) := p - p_\infty = \frac{q}{1 - q}. \quad \text{(3)}$$

Equation (2) is essentially the Martingale Condition of FGLW.
Relation between order size distribution and $p_\infty$

- Denote the (informed) order size required to consume limit orders up to price $p$ by $L_p = \sum_{k=0}^{p} \nu_p$ and the tail distribution function of metaorder sizes by $\tilde{F}(L) = P(\ell \geq L)$.

- Then the continuation probability $q$ is given by

$$q = P(\ell \geq L_p | \ell \geq L_{p-1}) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1})}.$$

- Thus

$$\Delta(p) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)}.$$  \hspace{1cm} (4)
Remarks

- Equation (4) relates the magnitude of transient impact to the distribution of metaorder sizes.
- (4) is valid for any given metaorder size distribution, not just for the power-law case explored in the following.
- Note that the price must revert immediately to $p_\infty$ after completion if $\mathcal{M}$ can tell when the metaorder ends.
  - In real markets, such information is partial and is inferred from observations; the price after completion decays over time to $p_\infty$. 

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In real markets, such information is partial and is inferred from observations; the price after completion decays over time to $p_\infty$. 

Suppose further that $M$ sets his prices so as to target zero expected profit for any order size. Then

$$L_p p_\infty = \sum_{k=0}^{p} k v_k$$  \hspace{1cm} (5)$$

This condition is not strictly imposed by the perfect competition assumption but can be obtained by supposing for example that $M$ is risk averse and minimizes his P&L volatility.

Equation (5) is the Fair Pricing condition of FGLW.

However (5) follows from the assumption of perfect competition between market makers. There is no notion of fair pricing here.
Price reversion

- The Fair Pricing Condition (5) gives

$$\Delta(p) = p - p_\infty = p - \frac{1}{L_p} \sum_{k=0}^{p} k v_k$$  \hspace{1cm} (6)

which may be rewritten as

$$\Delta(p) = \frac{1}{L_p} \sum_{k=0}^{p-1} L_k.$$ \hspace{1cm} (7)

- In particular, since $0 < L_k < L_p$ for all $k < p$, we must have $0 < \Delta(p) < p$.
  - That is, the price always reverts after completion, no matter what the distribution of metaorder sizes is.
Rôle of the bid-ask spread

- Equation (5) applied to the case \( p = 0 \) gives \( p_\infty = 0 \).
- The martingale condition then implies, as in FGLW, that the price cannot move.
  - This problem may be resolved by introducing a bid-ask spread.
- From (2), when the spread is a half-tick, zero expected profit per share imposes that
  \[
  \frac{1}{2} = (1 - q) p_\infty + q (p + 1) = q
  \]
  when \( p = p_\infty = 0 \). This then gives us the condition that
  \[
  \mathbb{P}(\ell \geq L_0) = \frac{1}{2}
  \]
  which fixes \( L_0 \) in terms of the spread.
- In FGLW, \( L_0 \) is an undetermined parameter which is used to set the scale of market impact. In contrast, the perfect competition assumption imposes a connection between \( L_0 \) and the spread.
The latent order book

- The volume $v_p$ that the market maker posts at price $p$ can be interpreted as the latent volume that would emerge were the price to reach $p$.

- In the Donier model, the shape of the latent order book reflects the adaptive reaction of the market-maker under perfect competition to the distribution of metaorder sizes, as estimated for example from the order flow autocorrelation function.

  - In particular, according to the LMF order-splitting model, if $P(\ell > L) \sim L^{-\gamma}$, the autocorrelation function of order flow decays as

$$\rho(\tau) \sim \frac{1}{\tau^{\gamma-1}}.$$
Recursion for $L_p$

Equation (4) reads

$$
\Delta(p) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)}
$$

and equation (7) reads

$$
\Delta(p) = \frac{1}{L_p} \sum_{k=0}^{p-1} L_k.
$$

Equating these two gives

$$
\frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)} = \frac{1}{L_p} \sum_{k=0}^{p-1} L_k.
$$

from which the $L_p$ may be computed recursively starting with (e.g.) $\tilde{F}(L_0) = 1/2$. 
Solving for the metaorder impact profile

- The latent order book is then given by
  \[ v_p = L_p - L_{p-1}. \]

- Plotting \( p \text{ vs } L_p \) gives the impact profile prior to completion.
- The reversion level is given by
  \[ \Delta(p) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)} \]
  where \( p \) is the maximum price reached.

- We can generate the impact profile for any choice of \( \tilde{F}(\cdot) \).
Example: The zeta distribution

Suppose that the probability of an order of length \( \ell \) is given by

\[
f(\ell) = \frac{\ell^{-\gamma+1}}{\zeta(\gamma + 1)}
\]

Then

\[
\tilde{F}(\ell) = \sum_{j=\ell}^{\infty} f(j) = \frac{\zeta(\gamma + 1, \ell)}{\zeta(\gamma + 1)} \sim \ell^{-\gamma} \text{ as } \ell \to \infty.
\]

Set \( \gamma = 3/2 \). The condition \( \tilde{F}(L_0) = 1/2 \) gives

\[
L_0 = 1.40541.
\]

We then find recursively

\[
\{L_1, L_2, L_3, \ldots\} = \{2.37, 3.43, 4.59, \ldots\}.
\]
Graphs of $L_p$ and $v_p$

Figure 4: Plots of the cumulative latent order depth $L_p$ (left) and the latent order density $v_p$ (right) vs $p$

- $L_p$ is roughly quadratic and $v_p = \Delta L_p$ is roughly linear.
The reversion level

Figure 5: Graph of the relative reversion $\alpha(p) := \Delta(p)/p$ vs $p$
The impact profile: Power-law case

Figure 6: The metaorder impact profile for a trade of length $L_{300}$

Market maker knows when metaorder has ended

Market maker guesses when metaorder ends

$p$

$L_p$
The deterministic version of the Donier model gives us the impact profile prior to completion.

- If $\mathcal{M}$ can tell when the metaorder has ended, the decay to $p_\infty$ is instant.
- Otherwise, we need to consider the information set available to $\mathcal{M}$. We will return to this later.
Asymptotic analysis: Power-law distribution

Rewrite (4) as

$$\frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1})} = \frac{1}{1 + \frac{1}{\Delta(p)}}.$$

Assume $\alpha(p) := \Delta(p)/p \rightarrow \alpha_{\infty} \in [0, 1]$ as $p \rightarrow \infty$. Taking logs gives

$$\log \tilde{F}(L_p) - \log \tilde{F}(L_{p-1}) = -\log \left(1 + \frac{1}{\alpha_{\infty} \cdot p}\right)$$

Then

$$\log \tilde{F}(L_p) = - \sum_{k=1}^{p} \log \left(1 + \frac{1}{\alpha_{\infty} \cdot p}\right) \sim -\frac{1}{\alpha} \log p \text{ as } p \rightarrow \infty \quad (10)$$
The power-law case: Impact profile

If $\tilde{F}(L) \sim L^{-\gamma}$, (10) gives

$$p(L) \sim L^{\alpha_\infty \gamma} \text{ as } L \to \infty$$

- This gives the impact profile prior to completion (asymptotically for large $L$).

The Fair Pricing condition (5) can be approximated for large $p$ as

$$p_\infty = \frac{1}{L_p} \int_0^{L_p} p(L) dL = \frac{p}{\alpha_\infty \gamma + 1}$$

Also, $p - \Delta(p) = (1 - \alpha_\infty) p$ so

$$\alpha_\infty = 1 - \frac{1}{\gamma} \text{ and } p_\infty = \frac{1}{\gamma} p.$$
Since $p \sim L^{\gamma^{-1}}$, we have

$$v_p = L_p - L_{p-1} \sim p^{\frac{1}{\gamma-1}} - (p - 1)^{\frac{1}{\gamma-1}}$$

$$\sim \frac{1}{\gamma - 1} p^{\frac{1}{\gamma-1} - 1}$$

With $\gamma \approx 3/2$ consistent with one of the stylized facts, we obtain both

- the linear latent order book of [Tóth, Bouchaud et al.], and
- the metaorder impact profile of FGLW, in particular the reversion level $p_\infty = \frac{2}{3} p$. 


Asymptotic profile vs exact profile

**Figure 7:** The blue points are the exact solution (9); the red dashed line is the asymptotic solution (10).

- The asymptotic solution is qualitatively very close to the exact solution.
Asymptotic analysis: Exponential distribution

Again assume $\alpha(p) := \Delta(p)/p \to \alpha_\infty \in [0, 1]$ as $p \to \infty$. Then if $\tilde{F}(L) \sim e^{-\lambda L}$, (10) gives

$$-\lambda L_p \sim \log \tilde{F}(L_p) = - \sum_{k=1}^{p} \log \left(1 + \frac{1}{\alpha_\infty p}\right) \sim -\frac{1}{\alpha} \log p \text{ as } p \to \infty$$

Equation (7) then reads

$$\alpha_\infty \sim \frac{1}{p} \frac{1}{L_p} \sum_{k=0}^{p-1} L_k = \frac{1}{p \log p} \sum_{k=0}^{p-1} \log k \to 1 \text{ as } p \to \infty$$

so the price reverts asymptotically to zero.

- This impact profile is completely inconsistent with empirical observation.
- We now confirm this with a numerical computation.
The impact profile: Exponential case

**Figure 8:** The metaorder impact profile for a trade of length $L_{300}$
Comparison with empirical impact profile

- Obviously the power-law profile is much more realistic.
- The convex impact in Figure 8 is completely inconsistent with the empirical impact profile of for example [Moro et al.] (Figure 3).
Price reversion

- As indicated earlier, if $\mathcal{M}$ does not know if the order has ended, the price cannot suddenly revert to $p_\infty$.
- However $\mathcal{M}$ knows both the participation rate $\pi$ of $\mathcal{I}$ and the number $n$ of informed child orders so far.
- Denote the event that the metaorder is still active by $\mathcal{A}$.
- Let $m$ be the number of uniformed child orders since the last informed order and assume that orders arrive as Poisson processes.

$\mathcal{M}$ may then use a Bayesian argument to compute $\mathbb{P}(\mathcal{A}|m)$ (it being implicit that we assume $n$ informed orders already observed).
A Bayesian argument

We have

\[ P(m|A) = (1 - \pi)^m \text{ and } P(m|\bar{A}) = 1. \]

Also, note that

\[ P(A) = \frac{\tilde{F}(n+1)}{\tilde{F}(n)} \]

is the probability that there are more informed child order to come, we have

\[
P(m) = P(m|A) P(A) + P(m|\bar{A}) P(\bar{A})
\]
\[
= (1 - \pi)^m P(A) + (1 - P(A))
\]
\[
= 1 - P(A) [1 - (1 - \pi)^m].
\]
Then

$$P(A|m) = \frac{P(m|A)P(A)}{P(m)} = \frac{(1 - \pi)^m P(A)}{1 - P(A) [1 - (1 - \pi)^m]}.$$ 

Let $p_m$ denote the price after $m$ uninformed child orders. Perfect competition imposes that

$$p_m = P(A|m) p_A + P(\bar{A}|m) p_\infty$$

where $p_A$ is the price if the metaorder continues. In particular,

$$p = p_0 = P(A) p_A + P(\bar{A}) p_\infty = p_\infty + P(A) (p_A - p_\infty).$$
Then

\[ p_m = p_\infty + \mathbb{P}(A|m) (p_A - p_\infty) \]

\[ = p_\infty + \frac{\mathbb{P}(A|m)}{\mathbb{P}(A)} (p - p_\infty) \]

\[ = p_\infty + \frac{(1 - \pi)^m}{1 - \mathbb{P}(A) [1 - (1 - \pi)^m]} (p - p_\infty) \quad (11) \]

- \( p_0 = p \). The price is fair if no uninformed child orders are detected.
- \( p_m \to p_\infty \) as \( m \to \infty \). If no new informed child orders are ever detected, the fair price is \( p_\infty \).
- Decay from \( p \) to \( p_\infty \) is exponential but scale invariant:

\[ p_m - p_\infty \sim e^{-m/\bar{m}} (p - p_\infty) \text{ with } \bar{m} = 1/\pi. \]
Power-law case

If $\tilde{F}(L) \sim L^{-\gamma}$, then as $n \to \infty$,

$$P(A) = \frac{\tilde{F}(n+1)}{\tilde{F}(n)} \sim \left(1 + \frac{1}{n}\right)^{-\gamma} \sim 1 - \frac{\gamma}{n}.$$

Then

$$p_m = p_\infty + \frac{(1 - \pi)^m}{1 - P(A) \left[1 - (1 - \pi)^m\right]} (p - p_\infty)$$

$$\sim p_\infty + \frac{(1 - \pi)^m}{1 - \left(1 - \frac{\gamma}{n}\right) \left[1 - (1 - \pi)^m\right]} (p - p_\infty).$$

We plot this price reversion profile in Figure 9.
IBM example impact profile

Figure 9: Typical price reversion profile

- $\mathcal{M}$ maintains the price at $p$ until he is certain that the metaorder is no longer active. Then the price drops quickly to $p_\infty$. 
Numerical example: IBM

- Consider a trade of 1% of daily volume of IBM which is around 40,000 shares, executed as a VWAP over 30 minutes (≈ 1/13 days).
- Participation rate $\pi \approx 13\%$.
- Assume power-law distributed order sizes with $\gamma = 3/2$.
- Further assume that each child order is for 200 shares (roughly the average trade size). Then there will be 200 child orders in this metaorder.
- The tick size is $0.01$ on a price of $200$.

With these parameters, this stylized version of the Donier model predicts a peak price move of $0.52$ with reversion to $0.35$.

- Assuming daily volatility of 2%, the square-root formula estimates $0.40$ so the numbers are of the right order of magnitude.
**IBM example impact profile**

**Figure 10:** The impact profile for 1% of DV traded over 30 minutes

- Scale invariance means that the impact profile for the same quantity traded over a different time interval (such as 2 hours) is obtained by dilation of the time axis.
Price manipulation

- For each new metaorder then, the price reverts exponentially to $p_\infty$ after some characteristic time that is proportional to the completion time of the order.
  - The decay constant should be related to the precision of the estimator used by $\mathcal{M}$ to detect the end of the metaorder.
- In the power-law case, $p_\infty = \alpha_\infty p \sim L^{\gamma-1}$, so if $\gamma < 2$, there should be price manipulation in the sense of Huberman and Stanzl (see [Gatheral]).
  - An obvious strategy would be to buy $2N$ shares using metaorders of length $N$ separated by an interval long enough to allow for full decay of the price to the permanent level. Then sell back using a metaorder of length $2N$. 
Bare and renormalized impact profiles

- The impact profile we have drawn is the *bare* impact profile assuming that:
  - There are no other active metaorders,
  - $\mathcal{M}$ has no memory of previous metaorders.

- To be consistent with empirical studies such as that of [Moro et al.], we need the *renormalized* impact profile which corresponds to averaging unconditionally over a dataset of metaorders, ignoring the initial state of the market.
The renormalized impact profile

- In general, if a given trader $A$ submits a child order, that order will either be in the same direction or the opposite direction to the net of other active metaorders (denoted by $B$).

- Denote the bare impact function by $I(\cdot)$ and the renormalized impact function by $\bar{I}(\cdot)$. Then assuming that $B$ has already traded $L$ shares, the impact of a single $A$ child order is given by

  $$\Delta \bar{I} = I(L + 1) - I(L) \approx I'(L) \text{ (same sign)}$$

  $$\Delta \bar{I} = -\{I(L - 1) - I(L)\} \approx I'(L) \text{ (opposite sign)}.$$

- In general, the sign of $B$’s metaorders will change several times during the execution of $A$’s metaorder.
Let $\ell$ be the size of $A$'s metaorder and assume that the participation rates of $A$ and $B$ are equal. Then

$$\bar{I}(\ell) \approx \sum_{i=1}^{N_\ell} L_i \cdot l'(L_i)$$

assuming that $B$ trades $\sum_i L_i$ shares during $A$'s metaorder execution (of $\ell$ shares) and $B$'s order changes sign $N_\ell$ times.
Two limiting cases

Denote the typical size of $B$'s orders by $\bar{L}$. Then

- If $\ell \ll \bar{L}$, $N_\ell = 1$ and
  \[
  \bar{I}(\ell) \approx \ell I'(\ell) .
  \]

- If $\ell \gg \bar{L}$, $N_\ell = \ell/\bar{L}$ and
  \[
  \bar{I}(\ell) \approx N_\ell \bar{L} I'(\bar{L}) = \ell I'(\bar{L}) \propto \ell .
  \]
Two limiting cases: Power-law impact

In the power-law case $I(L) = C L^\delta$ with $\delta = \gamma - 1 \approx \frac{1}{2}$, we have

$$\bar{I}(\ell) \approx \begin{cases} 
\delta I(\ell) & \text{if } \ell \ll \bar{L} \\
\delta \left(\frac{\bar{L}}{\ell}\right)^\delta I(\ell) & \text{if } \ell \gg \bar{L}.
\end{cases}$$

- We see that renormalized market impact is always less than bare market impact.
  - However, permanent impact is always nonzero.
- In the limit $\ell \gg \bar{L}$, market impact $\bar{I}(\ell)$ is linear in $\ell$.
  - So, for metaorders executed over long timescales, price manipulation is not possible.
  - The ratio of $p_\infty/p$ is preserved ($2/3$ if $\delta = 1/2$).
The Donier model framework has the following nice properties:

- Both the Martingale Condition and (an approximate version of) the Fair Pricing Condition of FGLW follow from the assumption of perfect competition between market makers.
- It provides a natural interpretation of the latent order book of [Tóth, Bouchaud et al.].

We computed the difference between the bare and renormalized market impact functions.

Price manipulation is not possible for long trades. It is possible that price manipulation is not possible in general.

Further analysis is both possible and required ...


Jim Gatheral, No-dynamic-arbitrage and market impact, Quantitative Finance 10(7) 749–759 (2010).


