Fractional volatility models

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Outline of this talk

- Motivation for fractional models
- Empirical volatility statistics
- Fractional Brownian motion (fBm)
- Prior fractional models of volatility
- A data-driven stochastic volatility model: fBergomi
- Fractional Stein Stein and fractional SABR models
Motivation I: Better fitting stochastic volatility models

- Conventional stochastic volatility models generate volatility surfaces that are inconsistent with the observed volatility surface.
  - In stochastic volatility models, the ATM volatility skew is constant for short dates and inversely proportional to $T$ for long dates.
  - Empirically, we find that the term structure of ATM skew is proportional to $1/T^\alpha$ for some $0 < \alpha < 1/2$ over a very wide range of expirations.

- The conventional solution is to introduce more volatility factors, as for example in the DMR and Bergomi models.

- One could imagine the power-law decay of ATM skew to be the result of adding (or averaging) many sub-processes, each of which is characteristic of a trading style with a particular time horizon.
Fitting the term structure of ATM skew

- According to (3.21) of [The Volatility Surface], the term structure of ATM skew in a conventional one-factor stochastic volatility model is roughly proportional to

\[ \psi(\kappa, \tau) := \frac{1}{\kappa \tau} \left\{ 1 - \frac{1 - e^{-\kappa \tau}}{\kappa \tau} \right\}. \]

- In Figure 1, we show that this function cannot fit the empirically observed term structure of ATM skew but that adding another such term (as a proxy for adding another factor) generates an excellent fit.
Empirical SPX ATM skew term structure with fits

**Figure 1:** The points in black are SPX ATM skews as of Sep 15, 2011. The red line is the best fit of $A \psi(\kappa, \tau)$. The blue line is the best fit of $A_1 \psi(\kappa_1, \tau) + A_2 \psi(\kappa_2, \tau)$. 
Bergomi Guyon

- Define the forward variance curve \( \xi_t(u) = \mathbb{E}[v_u | \mathcal{F}_t] \).
- According to [Bergomi and Guyon], in the context of a variance curve model, implied volatility may be expanded as

\[
\sigma_{BS}(k, T) = \sigma_0(T) + \sqrt{\frac{w}{T}} \frac{1}{2w^2} C^x \xi k + O(\eta^2) \quad (1)
\]

where \( \eta \) is volatility of volatility, \( w = \int_0^T \xi_0(s) \, ds \) is total variance to expiration \( T \), and

\[
C^x \xi = \int_0^T dt \int_t^T du \, \mathbb{E}[dx_t \, d\xi_t(u)].
\]
ATM volatility and the autocorrelation of volatility

- We may write $\xi_t(u) \approx \beta \xi_t(t) + \epsilon$ where $\epsilon \perp \xi_t(t)$ and
  $$\beta = \frac{\text{cov}(\xi_t(u) \xi_t(t))}{\text{var}(\xi_t(t))} = \frac{\text{cov}(v_u, v_t)}{\text{var}(v_t)}$$
  which is just the variance autocorrelation $\rho_v(u - t)$.
- Then
  $$C^\xi \approx \mathbb{E} \left[ \mathbb{E} \left[ d x_t \, d \xi_t(t) \right] \right] \int_0^T dt \int_t^T du \rho_v(u - t).$$
- Thus, the ATM volatility skew
  $$\psi(T) := \partial_k \sigma_{BS}(k, T)|_{k=0} \sim \frac{1}{T^2} \int_0^T dt \int_t^T du \rho_v(u - t)$$
  which relates the term structure of ATM skew to the variance autocorrelation function.
The Bergomi model

- The $n$-factor Bergomi variance curve model reads:

$$\xi_t(u) = \xi_0(u) \exp \left \{ \sum_{i=1}^{n} \eta_i \int_{0}^{t} e^{-\kappa_i (t-s)} dW_s^{(i)} + \text{drift} \right \}.$$  

\[ (2) \]

- To achieve a decent fit to the observed volatility surface, and to control the forward smile, we need at least two factors.
  - In the two-factor case, there are 8 parameters.
- When calibrating, we find that the two-factor Bergomi model is already over-parameterized. Any combination of parameters that gives a roughly $1/\sqrt{T}$ ATM skew fits well enough.
  - Moreover, the calibrated correlations between the Brownian increments $dW_s^{(i)}$ tend to be high.
Tinkering with the Bergomi model

- The term structure of ATM skew is related to the term structure of the autocorrelation function.
- The autocorrelation function of volatility is driven by the exponential kernel in the exponent in (2).
- It’s tempting to replace the exponential kernels in (2) with one or more power-law kernels.
- In the single factor case, this would give a model of the form

\[ \xi_t(u) = \xi_0(u) \exp \left\{ \eta \int_0^t \frac{dW_s}{(t-s)^\gamma} + \text{drift} \right\} \]

or more generally

\[ \xi_t(u) = \xi_0(u) \exp \left\{ \eta \int_0^t K(t-s) \, dW_s + \text{drift} \right\} \]

where the kernel \( K(\tau) \) has a power-law singularity as \( \tau \to 0 \).
Conversely

- Suppose the true model were something like

\[ \xi_t(u) = \xi_0(u) \exp \left\{ \eta \int_0^t K(t - s) \, dW_s + \text{drift} \right\} \]

- Then, using a discrete Laplace transform, we could approximate the kernel as

\[ (t - s)^{-\gamma} \approx \sum_{i=1}^n \alpha_i \, e^{-\kappa_i (t-s)} \]

for some coefficients \( \alpha_i \).

- Then we would have the Bergomi model back (but with all Brownians perfectly correlated).
Power-laws from averaging: A toy example

The following example, adapted from [Comte and Renault], illustrates how power-law behavior can emerge from the averaging of short memory processes.

- Consider the following OU process \( X_t = \log \sigma_t \) say indexed by \( \kappa \):

\[
X_t(\kappa) = \int_0^t e^{-\kappa(t-s)} dW_s.
\]

Then \( X_t \sim N(0, \Sigma(\kappa)^2) \) with \( \Sigma(\kappa)^2 = \int_0^t e^{-2\kappa(t-s)} ds \).

- Consider a multiplicity of such processes with gamma-distributed \( \kappa \). Explicitly,

\[
p_\Gamma(\kappa) = \frac{\kappa^{\alpha-1} e^{-\kappa/\theta}}{\theta^\alpha \Gamma(\alpha)}
\]

for some \( \alpha > 0 \) and \( \theta > 0 \).
Then, the average $\bar{X} \sim N(0, \bar{\Sigma}^2)$ with

$$\bar{\Sigma}^2 = \int_0^\infty p_{\Gamma}(\kappa) \int_0^t e^{-2\kappa(t-s)} d\kappa \, ds = \int_0^t \frac{1}{[1 + 2 \theta (t - s)]^\alpha} ds$$

and

$$\bar{X}_t = \int_0^t \frac{dW_s}{[1 + \theta (t - s)]^{\alpha/2}}.$$

Thus, averaging short memory volatility processes (with exponential kernels) over different timescales can generate a volatility process with a power-law kernel.
Motivation II: Power-law scaling of the volatility process

- A separate but (presumably) related reason for considering fractional volatility models is that the time series of realized volatility exhibits power-law scaling.
- The Oxford-Man Institute of Quantitative Finance makes historical realized variance (RV) estimates freely available at http://realized.oxford-man.ox.ac.uk. These estimates are updated daily.
- Using daily RV estimates as proxies for instantaneous variance, we may investigate the time series properties of \( v_t \) empirically.
SPX realized variance from 2000 to 2014

Figure 2: KRV estimates of SPX realized variance from 2000 to 2014.
The variance of SPX RV

- Let \( v_t^R \) denote the realized variance of SPX on day \( t \).
- We consider two measures of variance of RV over the time interval \( \Delta^1 \):
  1. \( V_1(\Delta) := \langle (v_{t+\Delta}^R - v_t^R)^2 \rangle \)
  2. \( V_2(\Delta) := \langle (\log(v_{t+\Delta}^R) - \log(v_t^R))^2 \rangle \)
- We find that
  \[ V_2(\Delta) = A \Delta^{2H} \text{ with } H \approx 0.14 \text{ and } A \approx 0.38. \]

\( \langle \cdot \rangle \) denotes a sample average.
Stationarity of $\log(v^R_t)$

- Suppose $\log(v^R_t)$ is mean-square stationary for large $t$ (as we certainly believe).

- Then,

$$V_2(\Delta) = \langle (\log(v^R_{t+\Delta}) - \log(v^R_t))^2 \rangle \leq 4 M_2$$

where $M_2 = \var(\log v^R) < \infty$.

- If so, we must have $V_2(\Delta) \sim \text{const.}$ as $\Delta \to \infty$. Power-law scaling of $V_2(\Delta)$ can hold only up to some long timescale.

- A hand-waving estimate of this timescale using $A \tilde{\Delta}^{2H} \approx 4 M_2$ gives $\tilde{\Delta} \approx 24$ years.
**Figure 3**: Log-log plots of $V_1(\Delta)$ and $V_2(\Delta)$ respectively. $V_2(\Delta)$ wins!
Figure 4: Histograms of \( \log v_{t+\Delta}^R - \log v_t^R \) for various lags \( \Delta \); normal fit in red; \( \Delta = 1 \) normal fit scaled by \( \Delta^{0.14} \) in blue.
**Q-Q plots of** $(\log v_{t+\Delta}^R - \log v_t^R)$ **for various lags** $\Delta$.

**Figure 5**: Q-Q plots of $(\log v_{t+\Delta}^R - \log v_t^R)$ for various lags $\Delta$. 
Repeating this analysis for all 21 indices in the Oxford-Man dataset yields:

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<th>Slope = 2H</th>
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<tr>
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<td>FTSEMIB.rk</td>
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Correlogram and test of scaling

Figure 6: The LH plot is a conventional correlogram of RV; the RH plot is of $\phi(\Delta) := \langle \log(\text{cov}(\sigma_{t+\Delta}, \sigma_t) + \langle \sigma_t \rangle^2) \rangle$ vs $\Delta^{2H}$ with $H = 0.14$. The RH plot again supports the scaling relationship $V_2(\Delta) \propto \Delta^{2H}$. 
A natural model of realized volatility

- Distributions of differences in the log of realized variance are close to Gaussian.
  - This motivates us to model $\nu_t$ (and so also $\sigma_t = \sqrt{\nu_t}$) as a lognormal random variable.
- Moreover, the scaling property of variance of RV differences suggests the model:
  \[
  \log \nu_{t+\Delta} - \log \nu_t = 2\eta \left(W^H_{t+\Delta} - W^H_t\right)
  \]
  where $W^H$ is fractional Brownian motion.
Heuristic derivation of autocorrelation function

We assume that \( \sigma_t = \bar{\sigma}_t e^{\eta W^H_t} \). Then

\[
\text{cov} [\sigma_t, \sigma_{t+\Delta}] = \bar{\sigma}_t \bar{\sigma}_{t+\Delta} \left[ \exp \left\{ \frac{\eta^2}{2} \left( t^2 H + (t + \Delta)^2 H - \Delta^2 H \right) \right\} - 1 \right]
\]

\[
\sim \bar{\sigma}_t \bar{\sigma}_{t+\Delta} \exp \left\{ \frac{\eta^2}{2} \left( t^2 H + (t + \Delta)^2 H - \Delta^2 H \right) \right\} \quad \text{as } t \to \infty.
\]

Similarly,

\[
\text{var} [\sigma_t] \sim \bar{\sigma}_t^2 \exp \left\{ \eta^2 t^2 H \right\}.
\]

Thus

\[
\rho(\Delta) = \frac{\text{cov} [\sigma_t, \sigma_{t+\Delta}]}{\sqrt{\text{var} [\sigma_t] \text{var} [\sigma_{t+\Delta}]}} \sim \exp \left\{ -\frac{\eta^2}{2} \Delta^2 H \right\}.
\]
Model vs empirical autocorrelation functions

Figure 7: Here we superimpose the predicted functional form of $\rho(\Delta)$ (in red) on the empirical curve (in blue).
It is a widely-accepted stylized fact that the volatility time series exhibits long memory.

- For example [Andersen et al.] estimate the degree $d$ of fractional integration from daily realized variance data for the 30 DJIA stocks.
  - Using the DPH estimator, they find $d$ around 0.35 which implies that the ACF $\rho(\tau) \sim \tau^{2d-1} = \tau^{-0.3}$ as $\tau \to \infty$.
- Using the same DPH estimator on the Oxford-Man RV data we find $d = 0.48$. But our model (3) is different from that of [Andersen et al.]. In our case, $\rho(\tau) \sim \tau^{2H-1} = \tau^{-0.72}$ as $\tau \to \infty$.
- We see clearly from Figures 4 and 6 that the realized volatility series is “medium memory” with $H \approx 0.14 < 1/2$. 

Fractional Brownian motion (fBm)

- **Fractional Brownian motion (fBm)** \( \{ W_t^H ; t \in \mathbb{R} \} \) is the unique Gaussian process with mean zero and autocovariance function

\[
\mathbb{E} \left[ W_t^H W_s^H \right] = \frac{1}{2} \left\{ |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right\}
\]

where \( H \in (0, 1) \) is called the **Hurst index** or parameter.

- In particular, when \( H = 1/2 \), fBm is just Brownian motion.
  - If \( H > 1/2 \), increments are positively correlated.
  - If \( H < 1/2 \), increments are negatively correlated.
Representations of fBm

There are infinitely many possible representations of fBm in terms of Brownian motion. For example, with $\gamma = \frac{1}{2} - H$,

\[
W_t^H = C_H \left\{ \int_{-\infty}^{t} \frac{dW_s}{(t-s)^\gamma} - \int_{-\infty}^{0} \frac{dW_s}{(-s)^\gamma} \right\}.
\]

where the choice

\[
C_H = \sqrt{\frac{2\,H\,\Gamma(3/2 - H)}{\Gamma(H + 1/2) \, \Gamma(2 - 2\,H)}}
\]

ensures that

\[
\mathbb{E} \left[ W_t^H W_s^H \right] = \frac{1}{2} \left\{ t^{2H} + s^{2H} - |t - s|^{2H} \right\}.
\]
Another representation of fBm

Define

\[ K_H(t, s) = C_H F\left(\gamma, -\gamma, 1 - \gamma, 1 - \frac{t}{s}\right) \frac{1}{(t-s)^\gamma}. \]

where \( F(\cdot) \) is Gauss’s hypergeometric function. Then, fBm can also be represented as:

\[ W^H_t = \int_0^t K_H(t, s) \, dW_s. \]

- The Mandelbrot-Van Ness representation uses the entire history of the Brownian motion \( \{W_s; s \leq t\} \).
- The Molchan-Golosov representation uses only the history of the Brownian motion from time 0.
Why “fractional”?

Denote the differentiation operator $\frac{d}{dt}$ by $D$. Then

$$D^{-1}f(t) = \int_0^t f(s) \, ds.$$ 

The Cauchy formula for repeated integration gives for any integer $n > 0$,

$$D^{-n}f(t) = \int_0^t \frac{1}{n!} (t - s)^{n-1} f(s) \, ds.$$ 

The generalization of this formula to real $\nu$ gives the definition of the fractional integral:

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - s)^{\nu-1} f(s) \, ds.$$ 

Note in particular that $D^0 f(t) = f(t)$. 
Comte and Renault: FSV model

[Comte and Renault] were perhaps the first to model volatility using fractional Brownian motion.

In their fractional stochastic volatility (FSV) model,

\[
\frac{dS_t}{S_t} = \sigma_t \, dZ_t
\]
\[
d \log \sigma_t = -\kappa (\log \sigma_t - \theta) \, dt + \gamma \, d\hat{W}^H_t
\]

(4)

with

\[
\hat{W}^H_t = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} \, dW_s, \quad 1/2 \leq H < 1
\]

and \( \mathbb{E}[dW_t \, dZ_t] = \rho \, dt \).

The FSV model is a generalization of the Hull-White stochastic volatility model.
Integral formulation

Solving (4) formally gives

\[ \sigma_t = \exp \left\{ \theta + e^{-\kappa t} (\log \sigma_0 - \theta) + \gamma \int_0^t e^{-\kappa (t-s)} d\hat{W}_s^H \right\}. \quad (5) \]

- \( H > 1/2 \) to ensure long-memory.
- Stationarity is achieved with the exponential kernel \( e^{-\kappa (t-s)} \)
  at the cost of introducing an explicit timescale \( \kappa^{-1} \).
FSV covariance

Define $y_t = \log \sigma_t$. We have

$$\text{cov}(y_t, y_{t+\Delta}) \propto \int_{-\infty}^{0} e^{\kappa s} ds \int_{-\infty}^{\Delta} e^{\kappa (s'-\Delta)} ds' \mid s - s' \mid^{2H-2}.$$ 

Then $\mathbb{E} \left[ (y_{t+\Delta} - y_t)^2 \right] = 2 \text{var}[y_t] - 2 \text{cov}(y_t, y_{t+\Delta})$ where

$$\text{cov}(y_t, y_{t+\Delta}) \propto \frac{e^{-k\Delta}}{2k^{2H}} \int_{0}^{k\Delta} \frac{e^{u} du}{u^{2-2H}} + \frac{e^{-k\Delta}}{2k^{2H}} \Gamma(2H - 1) + \frac{e^{k\Delta}}{2k^{2H}} \int_{k\Delta}^{+\infty} \frac{e^{-u} du}{u^{2-2H}}.$$
Vilela Mendes

- An empirical study of the scaling of volatility estimates by [Vilela Mendes and Oliveira] motivates their *data-reconstructed* model:

\[
\frac{dS_t}{S_t} = \sigma_t \, dZ_t
\]

\[
\log(\sigma_t) = \beta + \frac{k}{\delta} \left\{ W^H_t - W^H_{t-\delta} \right\}
\]

or equivalently,

\[
\sigma_t = \exp \left\{ \beta + \frac{k}{\delta} \int_0^t 1_{s>t-\delta} \, dW^H_s \right\}.
\]

- We note that this model looks very similar to the Comte-Renault model (5).
  - The indicator function kernel acts like the exponential kernel in FSV to force stationarity for long times.
Heston model

- The perennially popular and useful Heston model is given by:

\[
\frac{dS_t}{S_t} = \sqrt{v_t} \, dZ_t \\
dv_t = -\kappa (v_t - \bar{v}) \, dt + \eta \sqrt{v_t} \, dW_t
\]

with \(E[dW_t \, dZ_t] = \rho \, dt\).

- The variance process \(v_t\) is a stationary process with:

\[
E[v(t)] = \bar{v} \\
\text{var}[v_t] = \frac{\eta^2 \, \bar{v}}{2 \kappa} \\
\text{cov}[v_{t+h} \, v_t] = \frac{\eta^2 \, \bar{v}}{2 \kappa} e^{-\kappa |h|}.
\]

- The variance autocorrelation function is a decaying exponential and so the variance process is short memory.
Comte, Coutin and Renault: Affine model

In the affine fractional stochastic volatility (AFSV) model of [Comte, Coutin and Renault], the Heston model is extended by writing

\[
\frac{dS_t}{S_t} = \sqrt{v_t} \, dZ_t
\]

\[
v_t = \theta + \int_{-\infty}^{t} \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \, y_s \, ds, \quad 0 < \alpha < 1/2
\]

where \(y_t\) solves the CIR SDE

\[
dy_t = -\kappa (y_t - \bar{v}) \, dt + \eta \sqrt{y_t} \, dW_t.
\]

The constant \(\theta\) is introduced to break the otherwise tight connection between the mean and the variance of \(v_t\).
Autocovariance computations

- From the definition (8), \( v_t \) is a “fractionally integrated” CIR process.
- As in the Heston model, \( v_t \) is a stationary process with:

\[
\text{cov}[v_{t+h} v_t] = \frac{\eta^2 \bar{v}}{2 \kappa} \frac{\Gamma(1 - 2 \alpha)}{\Gamma(1 - \alpha) \Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\kappa |u|} |u + h|^{2\alpha - 1} \, du
\]

\[
\sim \frac{1}{\Gamma(2 \alpha)} (\kappa h)^{2\alpha - 1} \quad \text{as } h \to \infty
\]

– asymptotic power-law decay of the autocorrelation function.
Inherited Markovianity in the AFSV model

One might expect that expected realized variance (integrated variance) would depend on the whole history \( \{v_s; s < t\} \) of past instantaneous variance.

However, [Comte, Coutin and Renault] show that the unexpected component

\[
\tilde{w}_t(T) := \int_t^T v_s \, ds - \mathbb{E} \left[ \int_t^T v_s \, ds \bigg| \mathcal{F}_t \right]
\]

of the integrated variance depends on \( \mathcal{F}_t \) only through the current state \( v_t \).

Moreover, \( \mathbb{E} \left[ \int_t^T v_s \, ds \bigg| \mathcal{F}_t \right] \) is nothing other than the variance swap curve as of time \( t \).
Long memory vs realized variance (RV) data

- Notwithstanding that long memory of volatility is widely accepted as a stylized fact, RV data does not have this property.
- In Figure 8 we demonstrate graphically that existing long memory volatility models with $H > 1/2$ are not compatible with the RV data.
  - In the FSV model for example, the autocorrelation function $\rho(\tau) \propto \tau^{2H-2}$. Then, for long memory, we must have $1/2 < H < 1$.
    - For $\Delta \gg 1/\kappa$, stationarity kicks in and $V_2(\Delta)$ tends to a constant as $\Delta \to \infty$.
    - For $\Delta \ll 1/\kappa$, the exponential decay in (5) is not significant and $V_2(\Delta) \propto \Delta^{2H}$. 
Incompatibility of long-memory models with RV time series

Figure 8: Long memory models such as Comte-Renault (CR) and Vilela Mendes-Oliveira (VMO) are not compatible with the RV data. The blue line is CR with $k = 0.5$ and $H = 0.53$; the pink line is VMO with $\delta = 1$ day and $H = 0.8$. 
The fractional story

Suppose $\log \sigma_t =: y_t$ is given by a Gaussian Volterra process (in the terminology of [Decreusefond]) so that

$$y_t = \eta \int_{-\infty}^{t} K(t, s) \, dW_s \text{ with } K(t, s) \sim \frac{1}{(t-s)^\gamma} \text{ as } s \to t,$$

Then

$$y_u - y_t = \eta \int_{t}^{u} K(u, s) \, dW_s + \eta \int_{-\infty}^{t} [K(u, s) - K(t, s)] \, dW_s$$

$$=: \eta (M_{t,u} + Z_{t,u}).$$

Note that $\mathbb{E} [M_{t,u} | \mathcal{F}_t] = 0$ and $Z_{t,u}$ is $\mathcal{F}_t$-measurable.

- The point is that

$$\mathbb{E} [y_u | \mathcal{F}_t] - y_t = Z_{t,u}$$

and $\mathbb{E} [y_u | \mathcal{F}_t]$ is in principle computable from the volatility surface.
A data-driven stochastic volatility model

- Ignoring (at least for the time being) the difference between historical and pricing measures, we are led naturally from the data to the following model:

\[
\log v_u - \log v_t = 2 \eta (M_{t,u} + Z_{t,u}) \quad (9)
\]

- Integrating (9) then gives

\[
v_u = v_t \exp \left\{ 2 \eta M_{t,u} + 2 \eta Z_{t,u} \right\} = \mathbb{E} [v_u | \mathcal{F}_t] \exp \left\{ 2 \eta \int_t^u K(u, s) dW_s - 2 \eta^2 \int_t^u K(u, s)^2 ds \right\} \quad (10)
\]

- Of course this could be extended to \( n \) factors:

\[
v_u = \mathbb{E} [v_u | \mathcal{F}_t] \exp \left\{ 2 \int_t^u \sum_{i=1}^n \eta_i K^{(i)}(u, s) dW_s^{(i)} + \text{drift} \right\}
\]

- We could call this a fractional Bergomi or fBergomi model.
Features of the fractional Bergomi model

- The forward variance curve

\[ \xi_t(u) = \mathbb{E}[v_u | \mathcal{F}_t] = v_t \exp \left\{ Z_{t,u} + 2 \eta^2 \int_t^u K(u,s)^2 \, ds \right\}. \]

depends on the historical path \( \{W_s; s < t\} \) of the Brownian motion since inception (\( t = -\infty \) say).

- The fractional Bergomi model is non-Markovian:

\[ \mathbb{E}[v_u | \mathcal{F}_t] \neq \mathbb{E}[v_u | v_t]. \]

- However, given the (infinite) state vector \( \xi_t(u) \), which can in principle be computed from option prices, the dynamics of the model are well-determined.

- In practice, there is a bid-offer spread and we don’t have option prices for all strikes and expirations.
  - There is inherent model risk!
Re-interpretation of the conventional Bergomi model

- A conventional $n$-factor Bergomi model is not self-consistent for an arbitrary choice of the initial forward variance curve $\xi_t(u)$.
  - $\xi_t(u) = \mathbb{E}[\nu_t|\mathcal{F}_t]$ should be consistent with the assumed dynamics.

- Viewed from the perspective of the fractional Bergomi model however:
  - The initial curve $\xi_t(u)$ reflects the history $\{W_s; s < t\}$ of the driving Brownian motion up to time $t$.
  - The exponential kernels in the exponent of (2) approximate more realistic power-law kernels.

- The conventional two-factor Bergomi model is then justified in practice as a tractable Markovian engineering approximation to a more realistic fractional Bergomi model.
From fBergomi to fSABR

- Variance is lognormal in the fBergomi model and thus volatility is also lognormal.
- Then rewrite (10) as

$$\sigma_u = \hat{\sigma}_t(u) \mathcal{E} \left( \eta \int_t^u K(u, s) \, dW_s \right)$$

(11)

where $\mathcal{E}(X)$ is the stochastic exponential of $X$ and the forward volatility is given by

$$\hat{\sigma}_t(u) = \mathbb{E} \left[ \sigma_u \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \sqrt{v_u} \mid \mathcal{F}_t \right] = \sqrt{\xi_t(u)} \exp \left\{ -\frac{\eta^2}{2} \int_t^u K(u, s)^2 \, ds \right\}.$$
From fBergomi to fSABR

- The formal solution of the fractional Bergomi model may then be written as

\[
S_T = S_0 \exp \left\{ \int_0^T \sigma_t \, dZ_t - \frac{1}{2} \int_0^T \sigma_t^2 \, dt \right\}
\]

\[
= S_0 \mathcal{E} \left( \int_0^T \hat{\sigma}_0(t) \mathcal{E} \left( \eta \int_0^t K(t, s) \, dW_s \right) \, dZ_t \right).
\]

- \( \hat{\sigma}_0(t) \) is typically a slowly-varying function of \( t \) so we may write

\[
S_T \approx S_0 \mathcal{E} \left( \bar{\sigma}(T) \int_0^T \mathcal{E} \left( \eta \int_0^t K(t, s) \, dW_s \right) \, dZ_t \right) \quad (12)
\]

with \( \bar{\sigma}(T)^2 = \frac{1}{T} \int_0^T \hat{\sigma}_0(t)^2 \, dt \).
The fractional SABR (fSABR) model

Setting $K(t, s) = K_H(t, s)$ (the Molchan-Golosov kernel), we identify (12) as the solution of the following fSABR model:

$$\frac{dS_t}{S_t} = \sigma_t \, dZ_t$$

$$\sigma_t = \bar{\sigma}(T) \mathcal{E} \left( \alpha \, W_t^H \right)$$

where $dW_t \, dZ_t = \rho \, dt$.

- fSABR is the natural fBm extension of the SABR model of [Hagan et al.] (with $\beta = 1$).
- If $H < 1/2$, the variance of volatility grows sublinearly generating a natural term structure of ATM volatility skew.
- The fSABR stochastic volatility model is non-Markovian, just like the fractional Bergomi model.
The fractional Stein and Stein (fSS) model

As an even more tractable alternative, consider the following model:

\[
\frac{dS_t}{S_t} = \sigma_t \, dZ_t
\]

\[
\sigma_t = \sigma_0 + \eta \, W_t^H.
\]

again with \(dW_t \, dZ_t = \rho \, dt\).

- In the fSS model, the volatility \(\sigma_t\) is a normal random variable.
- Again, if \(H < 1/2\), the variance of volatility grows sublinearly thus volatility is “mean-reverting”.
- The fSS model can be considered a toy version of the more realistic fSABR model where more quantities of interest are explicitly computable.
Simulation of fSS and fSABR models

- First, for each Monte Carlo path, generate the correlated Brownian increments $\Delta W_t$ and $\Delta Z_t$.
- Given the $\Delta W_t$, the $W_t^H$ are constructed, for example using the Cholesky decomposition method.
SPX smiles in the fSABR model

- In Figure 9, we show how the fSABR model generates very good fits to the SPX option market as of 04-Feb-2010, a day when the ATM volatility term structure happened to be flat.
- fSABR parameters were: $\bar{\sigma} = 0.235$, $\eta = 0.2/0.235$, $H = 1/4$, $\rho = -0.7$.
  - Note in particular that we have obtained a good fit to the whole volatility surface using a model with very few parameters!
  - Moreover, $H$ and $\eta$ can be fixed from the VIX market.
fSABR fits to SPX smiles as of 04-Feb-2010

Figure 9: Red and blue points represent bid and offer SPX implied volatilities; orange smiles are the fSABR fit.
Consider the following approximation to ATM VIX volatility:

\[
\sigma_{VIX}^\text{BS} (t) \approx \eta \sqrt{\frac{t}{t + \epsilon}} \left[(t + \epsilon)^{2H} - \frac{\epsilon^{2H}}{c_H^{2H}}\right] \tag{13}
\]

where \( \epsilon = \Delta/2 \).

Up to a constant factor, the approximation (13) is in practice very accurate.

The approximation (13) is useful for two reasons:

1. It gives intuition for how the market implied value of \( H \) is fixed by the term structure of ATM VIX volatility.
2. It is much easier to determine \( H \) using (13) than from calibration using Monte Carlo.
Numerical results

- Calibrating $H$ to the observed VIX term structure as of 04-Feb-2010 using (13) gave $H = 0.212$.
- We see clearly that $H$ is indeed fixed by the term structure of ATM VIX volatility (or possibly by some other measure of volatility of volatility).
- With $H = 1/4$ and $\eta = 0.52$, we obtain the fit to the VIX option smiles displayed in Figure 10.
Figure 10: Red and blue points represent bid and offer VIX implied volatilities as of 04-Feb-2010; Dark green smiles are generated from Monte Carlo simulation of the fSABR model. fSABR parameters were: $\bar{\sigma} = 0.235$, $\eta = 0.52$, $H = 1/4$. 
Fit to SPX again

- With $H$ and $\eta$ fixed by the VIX market, we only have $\rho$ left to achieve a fit to the SPX market.

- With such a low volatility of volatility (relative to $\eta = 0.85$ from the original fit shown in Figure 9), we are forced to set $\rho = -0.99$ which is close to its negative limit.

- We obtain the fit to the SPX market shown in Figure 11.
  - Obviously not as nice as Figure 9!
Figure 11: Red and blue points represent bid and offer SPX implied volatilities as of 04-Feb-2010; Orange smiles are generated from Monte Carlo simulation of the fSABR model. fSABR parameters were: $\bar{\sigma} = 0.235$, $\eta = 0.52$, $H = 1/4$, $\rho = -0.99$. 
Summary

- Variance of the log of realized variance exhibits clear power-law scaling.
  - Long memory of RV is rejected by the data.
- The resulting data-driven model (fBergomi) is a non-Markovian generalization of the Bergomi model.
  - The conventional Markovian Bergomi market model can be viewed as an accurate approximation to fBergomi.
- The fSABR model, a natural generalization of the SABR model, is a tractable approximation to the fBergomi model.
  - fSABR fits observed smiles and skews remarkably well.
  - The value of the Hurst exponent $H$ is fixed by the term structure of VIX at-the-money implied volatility.
Motivation

fBm

Prior literature

The fBergomi model

fSS and fSABR models

References


