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BROWNIAN FLOW ON A FINITE INTERVAL WITH JUMP BOUNDARY CONDITIONS

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Abstract. We consider a stochastic flow in an interval $[-a, b]$, where $a, b > 0$. Each point of the interval is driven by the same Brownian path and jumps to zero when it reaches the boundary of the interval. Assuming that a/b is irrational we study the long term behavior of a random measure μ_t, the image of a finite Borel measure μ_0 under the flow. We show that if μ_0 is absolutely continuous with respect to the Lebesgue measure then the time averages of the variance of μ_t converge to zero almost surely. We also prove that for an arbitrary finite Borel measure μ_0 the Lebesgue measure of the support of μ_t decreases to zero as $t \to \infty$ with probability one.

1. Introduction.

We start with a purely deterministic construction. Fix $a, b > 0$ and let $\omega : [0, \infty) \to \mathbb{R}$ be an arbitrary continuous path. For each $x \in (-a, b)$ let $X_t^x(\omega)$ be the trajectory of a particle that starts at x, is driven by ω for all $t \geqslant 0$, and jumps to zero every time it reaches the boundary of the interval $(-a, b)$. More precisely, define a sequence of boundary hitting times by

$$\tau_0^x(\omega) = \inf\{t \geqslant 0 \,:\, x + \omega(t) - \omega(0) \in \{-a, b\}\},$$
$$\tau_n^x(\omega) = \inf\{t > \tau_{n-1}^x \,:\, \omega(t) - \omega(\tau_{n-1}^x) \in \{-a, b\}\},\ n \in \mathbb{N},$$

and set

$$X_t^x(\omega) = \begin{cases} x + \omega(t) - \omega(0) & \text{for } 0 \leqslant t < \tau_0^x, \\ \omega(t) - \omega(\tau_n^x) & \text{for } \tau_n^x \leqslant t < \tau_{n+1}^x,\ n \in \mathbb{N}. \end{cases} \tag{1}$$

Every trajectory $X_t^x(\omega)$, $t \geqslant 0$, is right continuous with left limits. Notice also that our particles are "sticky": if $X_s^x(\omega) = X_s^y(\omega)$ for some s then $X_t^x(\omega) = X_t^y(\omega)$ for all $t \geqslant s$. This property leads naturally to the following equivalence relation on $(-a, b)$.

Definition 1. Let $x, y \in (-a, b)$. We say that $x, y \in (-a, b)$ belong to the same equivalence class and write $x \sim y$, if there are integers k and l such that $y - x = ka + lb$.

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It is obvious that if the starting points of two particles are not in the same equivalence class then these particles will never collide. It is also easy to show that if $x \sim y$ then there is a continuous path ω such that $X_t^x(\omega) = X_t^y(\omega)$ for all sufficiently large t. This fact can be seen from the construction in Section 3, which leads to the proof of a much stronger statement (see Lemma 3).

For each path ω the correspondence $x \mapsto X_t^x(\omega)$ defines a family of transformations $\Phi^t(\omega)$, $t \geqslant 0$, of the interval $(-a,b)$ into itself satisfying the equation $\Phi^{t+s}(\omega) = \Phi^t(\theta_s\omega)\Phi^s(\omega)$, where

$$\theta_s\omega(\cdot) = \omega(s+\cdot) \tag{2}$$

is the canonical left shift in the space of continuous paths. Hence, for each ω we have a flow $\{\Phi^t(\omega), t \geqslant 0\}$. From now on whenever ω is fixed we drop it from the notation.

Denote by $S(t)$ the image of $(-a,b)$ under Φ^t:

$$S(t) = \Phi^t(-a,b) = \{X_t^x : x \in (-a,b)\}.$$

The following elementary lemmas describe some topological properties of Φ^t (see Figure 1).

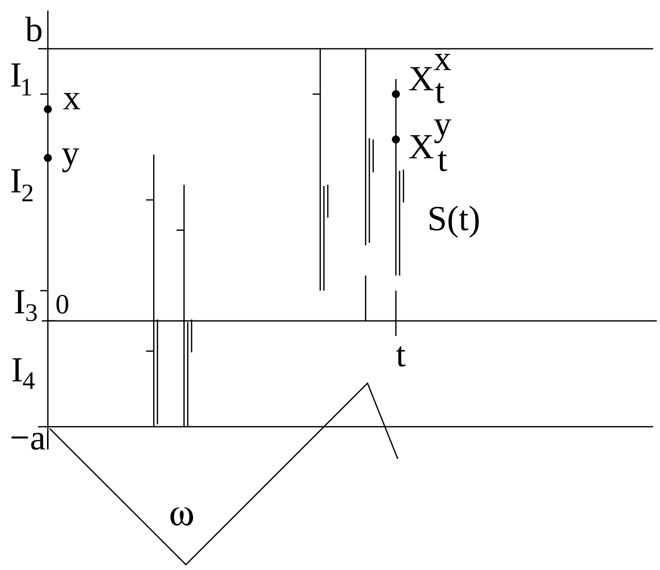

FIGURE 1. *Dynamics of* $S(t)$.

Lemma 1. *Fix an arbitrary continuous driving path* ω *and time* $t \geqslant 0$*. Let* $n(x)$ *be the number of times the trajectory* X_s^x, $s \in [0,t]$*, hits the boundary of* $(-a,b)$ *and* $N = \sup_{x\in(-a,b)} n(x)$*. Then*

(a) $N < \infty$;

(b) *there is a partition of* $(-a,b)$ *into finitely many intervals* I_i, $i = 1,2,\dots,k$, *such that if* $x \leqslant y$ *and* $[x,y] \subset I_i$ *for some* i *then*

$$X_t^y - X_t^x = y - x \quad \text{and} \quad \Phi^t([x,y]) = [X_t^x, X_t^y];$$

(c) $S(t)$ *is a disjoint union of finitely many intervals.*

A proof is given in the Appendix.

Lemma 2. *Fix an arbitrary continuous driving path ω. Let A and B be arbitrary Borel subsets of $(-a,b)$.*

(a) *If $A\subset B$ then $\Phi^t(A)\subset\Phi^t(B)$ for all $t\geqslant 0$.*

(b) *The Lebesgue measure of $\Phi^t(A)$ is a non-increasing function of t.*

This lemma is an obvious consequence of the definition of Φ^t, and we omit the proof.

Let μ_0 be a Borel probability measure on $(-a,b)$ and μ_t be the image of μ_0 under Φ^t, i.e. for each Borel set $A\subset(-a,b)$

$$\mu_t(A)=\mu_0((\Phi^t)^{-1}A)=\mu_0\{x\,:\,X_t^x\in A\}.$$

Lemma 1 ensures that μ_t is a Borel probability measure on $(-a,b)$. It also implies that if μ_0 is absolutely continuous with respect to Lebesgue measure λ then μ_t is also absolutely continuous with respect to λ for each $t\geqslant 0$, and $d\mu_t/d\lambda$ is well-defined (a.e.) by the formula

$$\frac{d\mu_t}{d\lambda}(y)=\begin{cases}\displaystyle\sum_{x:\,X_t^x=y}\frac{d\mu_0}{d\lambda}(x), & \text{if } y\in S(t)\\ 0, & \text{otherwise.}\end{cases}$$

In particular, if $\mu_0=\lambda$ then $d\mu_t/d\lambda$ is piecewise constant with values in $\{0,1,\dots,N\}$, $N=N(t,\omega)$ was defined in Lemma 1, and

$$\frac{d\mu_t}{d\lambda}\geqslant 1_{\{S(t)\}}\quad\text{a.e..}\tag{3}$$

Now we introduce a probability measure on paths $\omega\in\Omega=C([0,\infty)$. We make a canonical choice by letting $\mathcal{F}$ be the Borel σ-algebra on Ω and P be the Wiener measure on $(\Omega,\mathcal{F})$. Then the above construction defines a random flow $\{\Phi^t(\omega),t\geqslant 0\}$, $\omega\in\Omega$. Our goal is to study the long term behavior of random measures $\mu_t(\omega)$ when a/b is irrational. If a/b is rational the behavior of $\mu_t(\omega)$ changes dramatically. At the moment we do not have a satisfactory rigorous description of the asymptotic behavior of $\mu_t(\omega)$ for the rational case. Nevertheless several interesting results about geometric properties of $S(t)$ when a/b is rational were obtained by S. Athreya and S. Tanner [1].

This model was introduced by E. Cinlar at the Seminar on Stochastic Processes 2000. See [3] for motivation, the formal construction, and ergodic properties of the single particle process $\{X_t^x(\omega)\}_{t\geqslant 0}$. The probability of collision of two particles from the same equivalence class has been studied in [3] and [4].

Lemma 3 ([3],[4]). *Let $x\sim y$ and $\tau_{\rm col}=\inf\{t\geqslant 0\,|\,X_t^x=X_t^y\}$. Then*

$$P(\tau_{\rm col}<\infty)=1.$$

This result can also be seen directly from our construction in Section 3.

Notation. Without loss of generality we assume throughout the paper that

$$0<a<b \text{ and } a+b=1.\tag{4}$$

We write δ_x for the Dirac measure of mass 1 at x and λ for the Lebesgue measure on $(-a,b)$. The set of all non-negative integers is denoted by $\mathbb{Z}_+$, and $[x]$ stands for the integer part of $x\in\mathbb{R}$. P_x denotes the probability measure on continuous paths, which corresponds to the Brownian motion starting at x.

2. **Main results.** Let μ_0 be a Borel probability measure on $(-a,b)$ and $\mu_t(\omega)$, $t\geqslant 0$, be the random measure-valued process constructed in the previous section. Let

$$\bar{X}_t \equiv \int_{-a}^{b} x\,\mu_t(dx) = \int_{-a}^{b} X_t^x\,\mu_0(dx)$$

$$\sigma^2(t) \equiv \int_{-a}^{b} (x-\bar{X}_t)^2\,\mu_t(dx) = \int_{-a}^{b} (X_t^x-\bar{X}_t)^2\,\mu_0(dx).$$

be the mean and the variance of μ_t.

Theorem 1. *Assume that a/b is irrational and μ_0 is absolutely continuous with respect to the Lebesgue measure. Then*

$$\frac{1}{t}\int_0^t \sigma^2(s)\,ds \to 0 \quad P\text{-a.s. as } t\to\infty.$$

Moreover, for every $f\in C([-a,b])$ and for almost every $x\in(-a,b)$

$$\frac{1}{t}\int_0^t (f(X_s^x)-f(\bar{X}_s))^2\,ds \to 0 \quad P\text{-a.s. as } t\to\infty. \tag{5}$$

Theorem 2. *Assume that a/b is irrational. Then the Lebesgue measure of the support of μ_t converges to zero P-a.s as $t\to\infty$.*

The first statement of Theorem 1 says that the time averages of the variance of μ_t converge to zero. Roughly, this means that for very large t most of the time an observer would see one heavy "lump" about $\bar{X}_t$, which moves like a single particle plus some negligible "dust" on the rest of the interval. In Section 4 we give a simple example, which suggests that without the averaging in time Theorem 1 may not be true in general.

The proof of Theorem 1 is based on the consideration of the distance between two particles driven by the same Brownian path at the stopping times when one of them hits the boundary of the interval $(-a,b)$. We obtain an embedded Markov chain on the circle $S^1=\mathbb{R}/\mathbb{Z}$ and observe that it is generated by two automorphisms of S^1. We study the recurrence properties and stationary measures of this Markov chain and then apply Hopf's ratio ergodic theorem. The statements of Theorem 1 are immediate consequences of the results obtained for the embedded Markov chain.

Theorem 2 follows from Theorem 1 and Lemma 2. Due to the monotonicity property of the flow (see Lemma 2, part (a)) we do not need to assume the uniform continuity of μ_0 with respect to the Lebesgue measure.

Remark 1. We believe that the absolute continuity assumption in Theorem 1 can be removed.

The rest of the paper is organized as follows. In Section 3 we study an embedded Markov chain on the circle $S^1=\mathbb{R}/\mathbb{Z}$ and its stationary measures (Lemma 7). Using Hopf's ratio ergodic theorem we obtain Lemma 8 - the key ingredient in the proofs of Theorem 1 and Theorem 2. We derive these theorems in Section 4. The proofs of technical results are given in the Appendix.

3. **Embedded Markov chain.** In this section we construct a Markov chain whose properties are fundamental in establishing Theorem 1.

Consider the trajectories X_t^x and X_t^y, $x,y\in(-a,b)$, of two particles driven by the same Brownian path and record the changes in the distance between these particles for all $t \geqslant 0$. The distance between the two particles can change only when one of them hits the boundary $\{-a, b\}$. Since the waiting time of the first hit to the boundary is finite with probability one, we shall assume without loss of generality that $y = 0$.

Let $T_0 = 0$,

$$T_i = \inf\{t > T_{i-1} \,:\, X^x_{t-} \text{ or } X^0_{t-} \in \{-a, b\}\},\ i \in \mathbb{N}, \tag{6}$$

be the hitting times, and

$$\eta_i = T_{i+1} - T_i, \quad i = 0, 1, \ldots, \tag{7}$$

be the waiting times between two consecutive hits. Denote by $N(t)$ the number of hits of the boundary up to time t. That is

$$N(t) = \sup\{i \,:\, T_i \leqslant t\}. \tag{8}$$

When it does not lead to a confusion the dependence of T_i, η_i, and $N(t)$ on x will not be reflected in the notation.

The following lemma is an easy consequence of standard results about renewal processes, Brownian motion, and the law of large numbers. See Appendix for the proof.

Lemma 4. *The following statements hold with probability one:*

(i) $N(t) < \infty$ *for all* $t \geqslant 0$ *and* $N(t) \to \infty$ *as* $t \to \infty$*;*
(ii) *There is* $C_1 = C_1(\omega)$ *such that* $N(t)/t \leqslant C_1$ *for all* $t > 0$*;*
(iii) *There is* $C_2 = C_2(\omega)$ *such that* $\sum_{i=0}^n \eta_i^2/n \leqslant C_2$ *for all* $n \in \mathbb{N}$.

We will use the above lemma in the next section. Notice that either $X^x_{T_i} = 0$ or $X^0_{T_i} = 0$. Let $Z^x_i = X^x_{T_i} + X^0_{T_i}$, $i = 0, 1, \ldots$. In other words, Z^x_i is the position at time T_i of the particle, which is not at zero, unless they both are at zero, in which case $Z^x_i = 0$. The sequence $\{Z^x_i\}$, $i = 0, 1, \ldots$, is a Markov chain with the starting point x. We shall identify the ends of the interval $(-a, b)$, $a = 1 - b$, and consider this Markov chain on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Then $\{Z^x_i\}$ takes values in the set

$$S_x = \{y \in S^1 \,|\, y = x + kb\,(\text{mod } 1),\ k \in \mathbb{Z}\}, \tag{9}$$

which forms a closed class in the state space S^1. When b is rational, $b = m/n$, $(m, n) = 1$, S_x has n elements. When b is irrational S_x has infinitely many elements.

Before we write down the transition probabilities for the above Markov chain we would like to take a closer look at each step. Consider the stopping time T_i. Assume that $Z^x_i = y \neq 0$, which means that there was no collision before or at T_i. Then there are two possibilities for the next step: either the particle, which currently occupies 0, hits the boundary prior to the other particle or vice versa. The first case corresponds to the transition $y \to y + b\,(\text{mod } 1)$, and the second case corresponds to the transition $y \to b - y\,(\text{mod } 1)$. Recall that $a = 1 - b$. If the particles collided before or at time T_i then the process reached its absorbing state 0, and its further dynamics are the same as the dynamics for the single particle process.

Taking into account the preceding considerations, define two automorphisms R and Q of the circle S^1 by

$$Ry = y + b\,(\text{mod } 1),\ Qy = b - y\,(\text{mod } 1). \tag{10}$$

These maps have the following properties:

P1. Both R and Q preserve the Lebesgue measure on S^1.

P2. $RQRQ = QRQR = Q^2 = \mathrm{Id}$, where Id is the identity map on S^1.
P3. If b is irrational, then R is uniquely ergodic.

Properties P1 and P2 are straightforward, and P3 is a standard fact of the ergodic theory (see Section 1.9 of [6]).

The transition probabilities for our Markov chain $\{Z_i\}$, $i = 0, 1, \ldots$, are given by

$$\begin{aligned} &P(Z_1 = 0 \mid Z_0 = 0) = 1; \\ &P(Z_1 = Ry \mid Z_0 = y) = p(y) \text{ for } y \neq 0; \\ &P(Z_1 = Qy \mid Z_0 = y) = q(y) = 1 - p(y) \text{ for } y \neq 0; \end{aligned} \tag{11}$$

where

$$p(y) = \begin{cases} (b-y)/(1-y) & \text{if } 0 < y \leqslant b; \\ (y-b)/y & \text{if } b < y < 1. \end{cases} \tag{12}$$

Extend $p(y)$ periodically to $\mathbb{R}$. Notice that 0 is the only absorbing state and $p(b) = 0$, that is b always leads to 0. We shall also use the following fact: the function

$$h(y) = \begin{cases} (b-y)(1-b+y), & \text{for } 0 \leqslant y \leqslant b; \\ (y-b)(1+b-y), & \text{for } b < y < 1 \end{cases} \tag{13}$$

considered as a function on S^1 solves the equation

$$h(Ry)p(y) = h(y)p(Qy). \tag{14}$$

This is easily checked by taking the logarithms of both sides of (14) and substituting the formulas for p and h.

The main properties of $\{Z_i\}$ are summarized below.

Lemma 5. *Fix an arbitrary $x \in S^1$. Let $\{Z_i^x\}$, $i = 0, 1, \ldots$, be a Markov chain on S_x with the discrete topology and the transition probabilities (11). Then $\{Z_i^x\}$ has a stationary measure given by*

$$\varphi_x = \begin{cases} \delta_0 & \text{if } x \sim 0; \\ \sum_{y \in S_x} \rho(y)\delta_y, & \text{if } x \not\sim 0, \quad \text{where} \end{cases} \tag{15}$$

$$\rho(y) = \begin{cases} (y(b-y))^{-1} & \text{if } 0 < y < b; \\ ((1-y)(y-b))^{-1} & \text{if } b < y < 1. \end{cases} \tag{16}$$

If b is rational then φ_x is finite for all x, if b is irrational then φ_x is infinite for every $x \not\sim 0$.

Proof. The stationarity of φ_x is checked by a straightforward elementary computation, which we omitted. The last statement follows from the fact that $\rho(y)$ is bounded away from 0 on S^1. □

The next lemma studies the absorption and recurrence properties of $\{Z_i^x\}$.

Lemma 6. *For every $y \in S_0$ the absorption probability is equal to 1. Every state $y \in S_x$, $x \not\sim 0$, is positive recurrent if b is rational and null recurrent if b is irrational.*

Proof. If b is rational then S_0 is finite. Since 0 is the only absorbing state and the probability to reach 0 from every state $y \in S_0$ is positive, by a standard argument we conclude that the absorption probability for each $y \in S_0$ is equal to one (see [3], p. 840-841, for a detailed proof). If $x \not\sim 0$ then S_x is a finite irreducible closed class. Therefore, each state in S_x is positive recurrent (see, for example, [2], p. 293).

Assume now that b is irrational. Let $y \in S_x$, where $x \not\sim 0$. We shall apply Lemma 9. Re-enumerate the states of our chain by mappings $x + kb \mapsto k$ and $Q(x + kb) = (1 - k)b - x \mapsto k'$, $k \in \mathbb{Z}$ (see Figure 2 and Figure 4 below). Then we obtain a Markov chain studied in Lemma 9 with $p_k = p(x + kb)$ and $p_{k'} = p(Q(x + kb))$. According to Lemma 9 we only need to show the divergence of $\sum_{j=1}^{\infty} \beta_j q_j$ and $\sum_{j=1}^{\infty} \beta_{j'} q_{j'}$, where $q_j = 1 - p_j$, $q_{j'} = 1 - p_{j'}$, and β_j, $\beta_{j'}$ are given by (25). In our case q_j and $q_{j'}$ are bounded below by $(1 - b)$ (we assumed $b > 1/2$ in (4)). We shall show that there is a positive ε such that $\beta_j, \beta_{j'} > \varepsilon$ for infinitely many values of j. This will finish the proof of recurrence. By (14)

$$\beta_j = \prod_{k=1}^{j} \frac{p(Q(x + kb))}{p(x + kb)} = \prod_{k=1}^{j} \frac{h(R(x + kb))}{h(x + kb)} = \frac{h(x + (j + 1)b)}{h(x + b)}$$

Since b is irrational, for any $\varepsilon < 1/(4h(x + b))$ there are infinitely many values of j, for which β_j exceeds ε. Similar reasoning applies to the sequence $\{\beta_{j'}\}$.

The null recurrence follows from the fact that each S_x, $x \not\sim 0$, supports an infinite stationary measure φ_x defined by (15) (see [2], Chapter 5, Theorems 4.7 and 4.4).

Finally consider the case when $y \in S_0$ i.e. $y = kb \pmod 1$ for some integer k. To show that the absorption probability is equal to 1 we modify the chain considered in Lemma 9 by setting $0 = 0'$ and $p_0 = p_{0'} = 0$ and then follow our proof for recurrence, given in the previous paragraph. (See also [4].) □

Let us mention that Lemma 6 immediately implies Lemma 3.

Proof of Lemma 3. Consider the first time when one of the trajectories hits the boundary. This stopping time is finite with probability one. Starting from this stopping time we can apply our results about the embedded Markov chain. The relation $x \sim y$ implies that $\{Z_i\}$ starts from some state in S_0. By Lemma 6 the absorption probability is equal to 1. Also the waiting times between two consecutive hits are almost surely finite. The last two assertions imply the statement of the lemma. □

From now on we consider $\{Z_i\}$ on S^1 with its natural topology. Denote by $\mathcal{B}$ the Borel σ-algebra on S^1. Let $b \in \mathbb{R} \setminus \mathbb{Q}$. Observe that, unlike in the rational case, σ-finite measures φ_x, $x \not\sim 0$, of Lemma 5 are infinite for every Borel set B of positive Lebesgue measure (S_x is dense in S^1 and ρ is bounded away from 0). We have to find relevant stationary ergodic measures. Let $E_i = [1/i, b - 1/i] \cup [b + 1/i, 1 - 1/i]$, $i \in \mathbb{N}$. Then $S^1 = \bigcup_{i=1}^{\infty} E_i \cup \{0, b\}$. Define measure μ on $(S^1, \mathcal{B})$ as follows: $\mu(\{0, b\}) = 0$, and

$$\mu(B) = \lim_{i \to \infty} \int_{B \cap E_i} \rho(x)\, dx, \quad B \in \mathcal{B}.$$

In short, $\mu(dx) = \rho(x)\, dx$, where ρ is given by (16). Clearly, μ is σ-finite and is equivalent to λ.

Lemma 7. *Assume that b is irrational. Let $\{Z_i\}$, $i = 0, 1, \ldots$, be a Markov chain with the state space $(S^1, \mathcal{B})$ and the transition probabilities (11). Then μ is a stationary ergodic measure for $\{Z_i\}$ on $(S^1, \mathcal{B})$.*

Proof. To check that μ is stationary, it is enough to show that

$$\int_{S^1} f(x)\rho(x)\, dx = \int_{S^1} f(Rx)p(x)\rho(x)\, dx + \int_{S^1} f(Qx)q(x)\rho(x)\, dx.$$

for all $f \in C(S^1)$ such that $f, f \circ R$ and $f \circ Q$ are integrable on S^1 with respect to $\mu(dx)$. This is easily checked by substituting the expressions for p, q, ρ, and using the periodicity of f.

Now we show that μ is ergodic. Recall that measure μ on $(S^1, \mathcal{B})$ is said to be ergodic if for every invariant set (=closed class in our terminology) $B \in \mathcal{B}$ either $\mu(B) = 0$ or $\mu(B^c) = 0$. Let $B \in \mathcal{B}$ be an invariant set, and $\mu(B) > 0$. Then $\lambda(B) > 0$ (μ and λ are equivalent). If $x \in B$, $x \not\sim 0$, then $x + b \in B$, otherwise B would not be a closed class. Therefore, $B \subset R^{-1}B$ up to a set of measure 0. But R preserves λ, that is $\lambda(B) = \lambda(R^{-1}B)$. From the last two sentences we see that $B = R^{-1}B$ up to a set of measure 0. We conclude that B is an almost invariant set for R. Since $\lambda(B) > 0$, the ergodicity of R implies that $\lambda(B^c) = 0$. Thus, $\mu(B^c) = 0$, and μ is ergodic. □

Remark 2. Before proceeding any further we make several remarks and discuss the heuristics, which are at the core of our approach. Looking at the transition diagram (Figure 2) for our Markov chain we observe that

$$\begin{aligned} Z_i^x \in (b-\delta, b+\delta) \;&\Rightarrow\; Z_{i-1}^x \in (-\delta, \delta),\ i \geqslant 1 \\ Z_i^x \in (-\delta, \delta) \;&\Rightarrow\; Z_{i+1}^x \in (b-\delta, b+\delta),\ i \geqslant 0. \end{aligned} \tag{17}$$

Therefore, for every $x \notin S_0$ and at any time the number of visits of $\{Z_i^x\}$ to a δ-neighborhood of 0 can differ from the number of visits to a δ-neighborhood of b by at most 1.

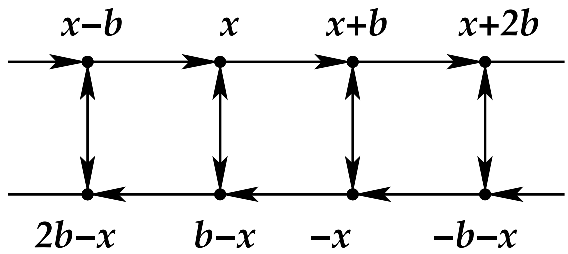

FIGURE 2. *Transition diagram of* $\{Z_i^x\}$.

Also if δ is small and Z_i^x is in the δ-neighborhood of b then with high probability (see (11) and (12)) Z_{i+1}^x will be in the δ-neighborhood of 0. Thus, a small neighborhood of the set $\Delta = \{0, b\}$ works as a "trap": the closer the particle gets to 0, the more time it spends in a small neighborhood of Δ before it leaves it. Outside a small neighborhood of Δ the function $p(x)$ is bounded away from zero. This suggests that, in the long run, for every starting point x the proportion of time spent in a small neighborhood of Δ should be close to 1. Combining this with our previous observation we conclude that the proportion of time spent near 0 is the same as the proportion of time spent near b and, therefore, should be close to 1/2. Consequently, we expect that for every $x \notin S_0$ and $f \in C(S^1)$ with probability 1

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} f(Z_i^x) = \frac{1}{2}(f(0) + f(b)). \tag{18}$$

Notice that the structure of the measure $\rho(x)dx$ reflects the above heuristics: the density ρ (extended periodically to $\mathbb{R}$) is symmetric with respect to $x = b/2$, bounded away from 0, and has poles at 0 and b. With this picture in mind we now turn to the last lemma of this section, which is the key ingredient of the proof of Theorem 1.

Consider the canonical infinite product space $\mathbb{X} = (S^1)^{\mathbb{N}}$ with the product σ-algebra $\mathcal{F} = \mathcal{B}^{\mathbb{N}}$. Let $\tilde{P}_x$ be the measure on $\mathbb{X}$ that corresponds to $\{Z_i^x\}$. Define

$$\tilde{P}_\mu = \int_{S^1} \tilde{P}_x \mu(dx), \tag{19}$$

where $\mu(dx) = \rho(x)dx$. Then $\tilde{P}_\mu$ is a σ-finite measure on $\mathbb{X}$, which is invariant under the left shift $\theta : \mathbb{X} \to \mathbb{X}$, $\theta(\mathbf{x}) = \theta(x_0, x_1, \dots) = (x_1, x_2, \dots)$. (Detailed construction can be adapted, for instance, from Section 2.5 of [5].)

Let $\Delta_\delta = (-\delta, \delta) \cup (b - \delta, b + \delta) \subset S^1$ and denote by $N_k^x(A)$ the number of visits of Z_i^x to the set A prior to time k,

$$N_k^x(A) = \sum_{i=0}^{k-1} 1_A(Z_i^x).$$

Lemma 8. *Assume that b is irrational. Then for an arbitrary $\delta < (1-b)/2$ and almost every $x \in S^1$ with $\tilde{P}_x$ probability one*

$$\lim_{k\to\infty} \frac{N_k^x(S^1 \setminus \Delta_\delta)}{k} = 0; \quad \lim_{k\to\infty} \frac{N_k^x((-\delta,\delta))}{k} = \lim_{k\to\infty} \frac{N_k^x((b-\delta, b+\delta))}{k} = \frac{1}{2}.$$

Proof. By Lemma 7 the measure μ is ergodic. This implies that $\tilde{P}_\mu$ is ergodic (see the proof of Theorem 6.9 in [8]). Hopf's ratio ergodic theorem for $(\mathbb{X}, \mathcal{F}, \tilde{P}_\mu, \theta)$ states that if $f, g \in L^1(\mathbb{X}, \tilde{P}_\mu)$, $g \geqslant 0$, and

$$\sum_{i=0}^{\infty} g(\theta^i \mathbf{x}) = \infty \quad \tilde{P}_\mu\text{-a.s.},$$

then

$$\lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} f(\theta^i \mathbf{x})}{\sum_{i=0}^{k-1} g(\theta^i \mathbf{x})} = \frac{\int_{\mathbb{X}} f \, d\tilde{P}_\mu}{\int_{\mathbb{X}} g \, d\tilde{P}_\mu} \quad \tilde{P}_\mu\text{-a.s.}. \tag{20}$$

Choosing f and g to depend only on the first coordinate of $\mathbf{x}$ it is not difficult to see from (19) and (20) that there exists a set $\mathcal{N}(f,g) \in \mathcal{B}$ with $\lambda(\mathcal{N}) = 1$ such that for every $x \in \mathcal{N}$

$$\lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} f(Z_i^x)}{\sum_{i=0}^{k-1} g(Z_i^x)} = \frac{\int_{S^1} f \, d\mu}{\int_{S^1} g \, d\mu} \quad \tilde{P}_x\text{-a.s.}.$$

Using the recurrence property of $\{Z_i^x\}$ and the ratio ergodic theorem with $f(x) = 1_{\{S^1 \setminus \Delta_\delta\}}(x)$, $g(x) = 1_{\{\Delta_\delta \setminus \Delta_\varepsilon\}}(x)$, $\varepsilon < \delta$, and then letting $\varepsilon \to 0$ we see that for a.e. x

$$\lim_{k\to\infty} \frac{N_k^x(S^1 \setminus \Delta_\delta)}{N_k^x(\Delta_\delta)} = 0 \quad \tilde{P}_x\text{-a.s.}. \tag{21}$$

Since $N_k^x(\Delta_\delta) \leqslant k$, (21) gives us the first of the required limits.

Relations (17) immediately imply that for all k and $x \notin S_0$

$$|N_k^x((-\delta,\delta)) - N_k^x((b-\delta, b+\delta))| \leqslant 1.$$

This fact together with (21) and the obvious equality

$$1 = \frac{N_k^x((-\delta,\delta))}{k} + \frac{N_k^x((b-\delta, b+\delta))}{k} + \frac{N_k^x(S^1 \setminus \Delta_\delta)}{k}.$$

completes the proof of the lemma. □

4. **Proofs of main results.**

Proposition 1. *Let a/b be irrational. Then for almost every pair $x, y \in (-a, b)$*

$$\frac{1}{t}\int_0^t (X_s^x - X_s^y)^2\, ds \to 0 \quad P\text{-a.s. } as\ t \to \infty.$$

Proof. If $x \sim y$ then by Lemma 3 there is nothing to prove. Without loss of generality we may assume that $x \not\sim y$, $x \in (-a, b)$ and $y = 0$. The proof is based on Lemma 4 and Lemma 8 from Section 3.

Let δ be an arbitrary small positive number. We consider Z_i^x to be a point on $S^1 = \mathbb{R}/\mathbb{Z}$ with the usual identification of $-a$ and b as in Section 3 and denote by $\|Z_i^x\|$ the distance on the real line from $Z_i^x \in (-a, b)$ to the origin. We obtain

$$\begin{aligned}\frac{1}{t}\int_0^t (X_s^x - X_s^y)^2\, ds &\leqslant \frac{1}{t}\sum_{i=0}^{N(t)} \|Z_i^x\|^2 \eta_i^x \\ &\leqslant \frac{1}{t}\sum_{i=0}^{N(t)} \|Z_i^x\|^2 \eta_i^x \left(1_{\{S^1\setminus\Delta_\delta\}}(Z_i^x) + 1_{\{(-\delta,\delta)\}}(Z_i^x) + 1_{\{(b-\delta,b+\delta)\}}(Z_i^x)\right).\end{aligned}$$

We split the sum into three parts and estimate each part separately. For each $k \in \mathbb{N}$ we get

$$\frac{1}{k}\sum_{i=0}^{k-1} |Z_i^x|^2 \eta_i^x 1_{\{S^1\setminus\Delta_\delta\}}(Z_i^x) \leqslant \left(\frac{1}{k}\sum_{i=0}^{k-1}(\eta_i^x)^2\right)^{1/2} \left(\frac{N_k^x(S^1\setminus\Delta_\delta)}{k}\right)^{1/2};$$

$$\frac{1}{k}\sum_{i=0}^{k-1} |Z_i^x|^2 \eta_i^x 1_{\{(-\delta,\delta)\}}(Z_i^x) \leqslant \delta^2 \left(\frac{1}{k}\sum_{i=0}^{k-1}(\eta_i^x)^2\right)^{1/2};$$

$$\begin{aligned}\frac{1}{k}\sum_{i=0}^{k-1} |Z_i^x|^2 \eta_i^x 1_{\{(b-\delta,b+\delta)\}}(Z_i^x) &\leqslant \frac{1}{k}\sum_{i=0}^{k-1}(\eta_i^x - E\eta_i^x) 1_{\{(b-\delta,b+\delta)\}}(Z_i^x) \\ &\quad + \frac{1}{k}\sum_{i=0}^{k-1} 1_{\{(b-\delta,b+\delta)\}}(Z_i^x) E\eta_i^x.\end{aligned}$$

By Lemma 4 and Lemma 8 for almost every x and for almost all Brownian paths the right hand sides of the first two inequalities are small for all sufficiently large k. Observe that $E\eta_i^x \leqslant \delta$ for all i such that $Z_i^x \in (b-\delta, b+\delta) \in S^1$. Therefore,

$$\begin{aligned}\frac{1}{k}\sum_{i=0}^{k-1} |Z_i^x|^2 \eta_i^x 1_{\{(b-\delta,b+\delta)\}}(Z_i^x) &\leqslant \frac{1}{k}\sum_{i=0}^{k-1}(\eta_i^x - E\eta_i^x) 1_{\{(b-\delta,b+\delta)\}}(Z_i^x) \\ &\quad + \delta\, \frac{N_k^x((b-\delta, b+\delta))}{k}.\end{aligned}$$

Since $E(\eta_i^x)^2 < C$, the law of large numbers and Lemma 8 imply that for almost every x the right hand side of the last inequality is also small P-a.s.. Finally we combine the above estimates with the first two parts of Lemma 4, and arrive at the desired conclusion. □

Proof of Theorem 1. The first statement follows from Proposition 1 by the integration over $(-a, b) \times (-a, b)$ with respect to the product measure $\mu_0(dx) \times \mu_0(dy)$ and by the Cauchy-Schwartz inequality.

To prove the second claim, choose an arbitrary $\varepsilon > 0$. Let $M = \max_{[-a,b]} |f|$. Since f is continuous on $[-a, b]$, there exists a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ and $x_1, x_2 \in [-a, b]$. We have

$$\begin{aligned}\frac{1}{t}\int_0^t (f(X_s^x) - f(\bar{X}_s))^2\, ds &= \frac{1}{t}\int_0^t (f(X_s^x) - f(\bar{X}_s))^2 1_{\{s:|X_s^x - \bar{X}_s|<\delta\}}\, ds \\ &+ \frac{1}{t}\int_0^t (f(X_s^x) - f(\bar{X}_s))^2 1_{\{s:|X_s^x - \bar{X}_s|\geqslant\delta\}}\, ds \\ &\leqslant \varepsilon^2 + \frac{4M^2}{t\delta^2}\int_0^t |X_s^x - \bar{X}_s|^2\, ds \\ &\leqslant \varepsilon^2 + \frac{4M^2}{t\delta^2}\int_0^t \int_{-a}^b (X_s^x - X_s^y)^2\, \mu_0(dy)\, ds,\end{aligned}$$

where we used Chebyshev's inequality and the Cauchy-Schwartz inequality. From Proposition 1 we immediately obtain the (5). □

Proof of Theorem 2. Since for every initial measure μ_0 the support of μ_t is contained in the closure of $S(t)$ and $S(t)$ is a finite collection of intervals of positive length (see Lemma 1), it is enough to prove that $\lambda(S(t))$ converges to zero as $t \to \infty$ (as taking the closure of a finite set of intervals will not change its Lebesgue measure). Therefore, for the proof below we assume that $\mu_0 = \lambda$ and μ_t is the image of λ under the flow.

By Lemma 2 $\lambda(S(t))$ is a non-increasing function of t. Therefore, $\lambda(S(t))$ has a (possibly random) limit as $t \to \infty$, which we denote by ξ. Fix an arbitrary $\varepsilon > 0$. Using (3) for we obtain

$$\begin{aligned}\sigma^2(t) = \int_{-a}^b (y - \bar{X}_t)^2\, \mu_t(dy) \geqslant \int_{S(t)} (y - \bar{X}_t)^2\, dy \geqslant \\ \varepsilon^2 \lambda\{y \in S(t),\ |y - \bar{X}_t| > \varepsilon\} \geqslant \varepsilon^2(\xi - 2\varepsilon).\end{aligned}$$

By Theorem 1

$$\varepsilon^2(\xi - 2\varepsilon) \leqslant \frac{1}{t}\int_0^t \sigma^2(s)\, ds \to 0, \quad \text{as} \quad t \to \infty \quad \text{P-a.s.},$$

which implies that $\xi = 0$ with probability one. □

Appendix A. **Appendix.** In this section we give proofs of Lemma 1, Lemma 2, and Lemma 4. We also state and prove a recurrence result that we used in Section 3.

Proof of Lemma 1. (a) Since ω is continuous, it is uniformly continuous on $[0, t]$. In particular, there is a $\delta = \delta(\omega, t) > 0$ such that $|\omega(r) - \omega(s)| < a$ for all $|r - s| < \delta$, $r, s \in [0, t]$. Then $n(0) \leqslant 1 + [t/\delta]$, and, hence, $n(x) \leqslant 2 + [t/\delta]$ for every $x \in (-a, b)$ (we have to allow for an additional hit when the particle starts close to the boundary).

(b) For each $x \in (-a, b)$ consider its trajectory X_s^x, $s \in [0, t]$. Record a sequence of boundary hits, $\overline{x}_t$, in chronological order by using the letter b for the upper boundary hits and letter a for the lower boundary hits.
Observe that if for $x < y$ the trajectories X_s^x and X_s^y, $s \in [0, t]$, have the same sequence, that is $\overline{x}_t = \overline{y}_t$ (see Figure 3), then $\overline{z}_t = \overline{x}_t$ for every $z \in (x, y)$. Thus,

$$\Phi^t([x, y]) = [X_t^x, X_t^y] = [X_t^x, X_t^x + y - x]. \tag{22}$$

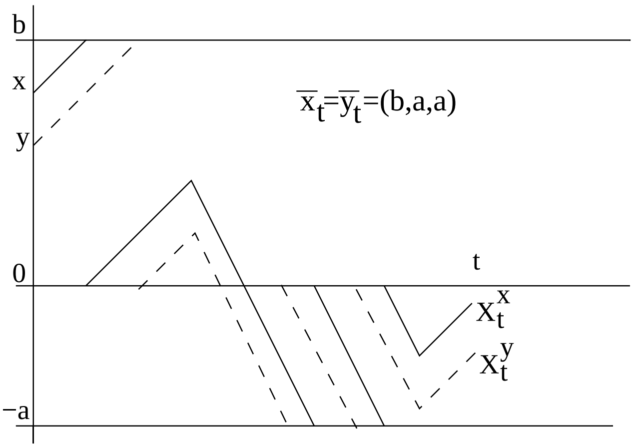

FIGURE 3. *Two trajectories with the same sequence of boundary hits.*

Define

$$I_x = \{y \in (-a, b) \,:\, \overline{y}_t = \overline{x}_t\}.$$

From (22) we conclude that for each $x \in (-a, b)$ the set I_x is an interval. Intervals I_x and I_y are either disjoint or the same. Therefore, all distinct I_x's form a partition of $(-a, b)$. Since the length of each sequence of boundary hits is bounded by N (see (a)) and the number of distinct two-letter sequences of length at most N is finite, we conclude that this partition is finite.

(c) From part (b) we know that $\Phi^t(I_x)$ is an interval (see also Figure 1). Since $S(t)$ is the union of $\Phi^t(I_x)$, $x \in \mathcal{I}$, where $\mathcal{I}$ is finite, the claim follows. □

Proof of Lemma 4. Assume that at time 0 we have two particles labelled with $k = 1, 2$, particle 1 starts from x and particle 2 starts from y.

Step 1. Consider each particle separately. Denote by $\eta_j^{(k)}$, $j \geqslant 0$, and $N^{(k)}(t)$ the waiting times between two consecutive hits of the boundary and the number of hits up to time t for the particle k respectively. The definitions are similar to (6)-(8) and are omitted. Then for each $k = 1, 2$ the statements of the lemma hold for $N^{(k)}(t)$ and the sequence $\eta_j^{(k)}$, $j \geqslant 0$. More precisely, parts (i) and (ii) are standard results about Brownian motion. The uniform boundedness of $N^{(k)}(t)/t$ on the interval $t > 0$ follows from the convergence of $N^{(k)}(t)/t$ to a finite limit as $t \to \infty$ and the fact that $N^{(k)}(t) = 0$ for all t smaller that the first hitting time $T_1^{(k)}$. Part (iii) is the consequence of the strong law of large numbers. See, for example, [2], Theorem 4.1, p. 204, Theorem 5.5, p. 399, and Theorem 5.9, p. 401.

Step 2. Consider both particles. The hitting times T_i, $i \geqslant 0$, for the pair of particles are obtained by ordering the union of the sets of hitting times for particles 1 and 2. Therefore,

$$N(t) \leqslant N^{(1)}(t) + N^{(2)}(t). \tag{23}$$

Observe also that for each $i \in \mathbb{N}$ there is a pair (j, k), $j \leqslant i$, and $k \in \{1, 2\}$, such that $T_i = T_j^{(k)}$. Moreover, the mapping $i \to (j, k)$ is injective. Then, obviously,

$\eta_i \leqslant T_{j+1}^{(k)} - T_j^{(k)} = \eta_j^{(k)}$, $j \leqslant i$. This implies that

$$\frac{1}{n}\sum_{i=0}^{n}\eta_i^2 \leqslant \frac{1}{n}\left(\sum_{j=0}^{n}(\eta_j^{(1)})^2 + \sum_{j=0}^{n}(\eta_j^{(2)})^2\right). \tag{24}$$

The statements of the lemma are now immediate consequences of Step 1 and relations (23) and (24). □

A recurrence result. Now we prove a lemma about recurrence that we used in Section 4. Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $\mathbb{Z}' = \{0', \pm 1', \pm 2', \dots\}$. Consider a Markov chain $\{X_i\}$, $i = 0, 1, \dots$, on $\mathbb{Z} \cup \mathbb{Z}'$ with the transition diagram

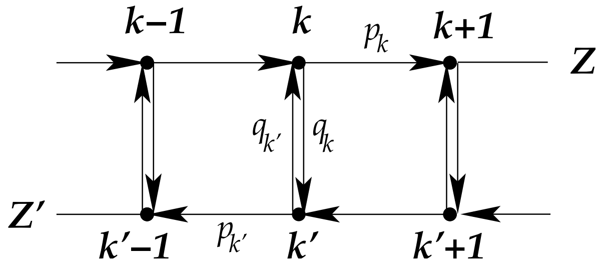

FIGURE 4. *Transition diagram for* $\{X_i\}$.

where $p_x, q_x \in (0,1)$ and $p_x + q_x = 1$ for all $x \in \mathbb{Z} \cup \mathbb{Z}'$. This chain is irreducible. Define

$$\beta_j = \prod_{k=1}^{j} \frac{p_{k'}}{p_k}, \quad \beta_{j'} = \prod_{k=-1}^{-j} \frac{p_k}{p_{k'}}. \tag{25}$$

Lemma 9. *Let $\{X_i\}$ be a Markov chain with transition diagram (4). Then $\{X_i\}$ is recurrent if and only if both series $\sum_{j=1}^{\infty} q_j\beta_j$ and $\sum_{j=1}^{\infty} q_{j'}\beta_{j'}$ diverge.*

Proof. Fix an arbitrary $n \in \mathbb{N}$. Let $T_x = \inf\{i \geqslant 0 : X_i = x\}$ and $a_n(x) = P_x(T_{0'} < T_n)$. Clearly, $a_n(0') = 1$ and $a_n(n) = 0$, and for every starting point $x = k$ or $x = k'$, $k \in \{0, 1, \cdots, n\}$ we have that $T_n \geqslant n-k$. Therefore, $\{T_{0'} < \infty\} = \cup_{n=1}^{\infty}\{T_{0'} < T_n\}$, and

$$\rho_{00'} \stackrel{\text{def}}{=} P_0(T_{0'} < \infty) = \lim_{n\to\infty} a_n(0).$$

We can define $\rho_{0'0}$ in a similar way. Since $\{X_i\}$ is irreducible, it is recurrent if and only if both $\rho_{00'}$ and $\rho_{0'0}$ are equal to 1 (see [2], Theorem 3.4).

Below we compute $a_n(0)$ and find $\rho_{00'}$. From the transition diagram we obtain

$$a_n(k) = a_n(k+1)p_k + a_n(k')(1 - p_k); \tag{26}$$

$$a_n(k') = a_n(k'-1)p_{k'} + a_n(k)(1 - p_{k'}). \tag{27}$$

Adding (26) and (27) gives

$$p_k a_n(k') + p_{k'} a_n(k) = p_k a_n(k+1) + p_{k'} a_n(k'-1),$$

which can be rewritten in a more convenient form and reiterated as follows

$$a_n(k+1) - a_n(k') = \frac{p_{k'}}{p_k}(a_n(k) - a_n(k'-1)) = \cdots = \beta_k(a_n(1) - 1), \tag{28}$$

Here we used the boundary condition $a_n(0') = 1$ and (25). From (26) we also get

$$a_n(k')q_k = a_n(k) - a_n(k+1)p_k. \tag{29}$$

Multiplying (28) by q_k, using (29) and the fact that $p_k + q_k = 1$ we arrive at the recursive relation

$$a_n(k+1) = a_n(k) + q_k\beta_k(a_n(1) - 1) = \cdots = a_n(1) + \sum_{j=1}^{k} q_j\beta_j(a_n(1) - 1).$$

Choosing $k = n - 1$ and applying the condition $a_n(n) = 0$ we can find $a_n(1)$ and, therefore, all $a_n(k)$, $k = 1, 2, \ldots, n-1$:

$$a_n(k) = \frac{\sum_{j=k}^{n-1} q_j\beta_j}{1 + \sum_{j=1}^{n-1} q_j\beta_j}.$$

Finally from (26) with $k = 1$ we get

$$a_n(0) = a_n(1)p_0 + (1 - p_0) = 1 - \frac{p_0}{1 + \sum_{j=1}^{n-1} q_j\beta_j}.$$

Taking the limit as $n \to \infty$ we find $\rho_{00'}$. A similar computation gives $\rho_{0'0}$. Thus,

$$\rho_{00'} = 1 - \frac{p_0}{1 + \sum_{j=1}^{\infty} q_j\beta_j}, \quad \rho_{0'0} = 1 - \frac{p_0'}{1 + \sum_{j=1}^{\infty} q_{j'}\beta_{j'}},$$

and the proof is now complete. □

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