Minimizer Graphs for a Class of Extremal Problems

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Abstract: We consider the family of graphs with a fixed number of vertices and edges. Among all these graphs, we are looking for those minimizing the sum of the square roots of the vertex degrees. We prove that there is a unique such graph, which consists of the largest possible complete subgraph plus only one other non-isolated vertex. The same result is proven for any power of the vertex-degrees less than one half.
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1. INTRODUCTION

In this paper, we study an extremal problem in graph theory: among all simple graphs on n vertices and e edges, find the ones where the sum of the square roots of the vertex-degrees is minimum. We call these graphs minimizer graphs.

This problem was first considered by Linial and Rozenman [4], in connection to certain isoperimetric inequalities for subsets of the discrete unit cube; see [4,7] for more details. In [4], it has been shown that, for the special case when $e = \binom{k}{2}$, there is a unique minimal graph consisting of a complete graph on k vertices and $n-k$ isolated vertices.

Linial and Rozenman have conjectured that among all graphs with e edges, $\binom{k-1}{2} < e < \binom{k}{2}$, the smallest sum is attained precisely when the only nontrivial component is the graph obtained from K_{k-1} by adding one new vertex of degree $e - \binom{k-1}{2}$.

In this paper, we prove this conjecture. We also prove that the same graphs are the unique (up to an isomorphism) minimizer graphs for the sum of powers α of the vertex degrees, for all $\alpha < 1/2$.

Throughout the paper, K_p will denote the complete graph with p vertices, S_p will be used for the edgeless graph with p vertices, and $K_{1,e}$ will indicate the star with e edges.

The rest of the paper is structured as follows. In Section 2, we discuss some preliminary results which will be used in section 3 when we prove our main result, the uniqueness of the minimizer graph for the sum of the square roots of the vertex degrees. In Section 4, we conclude our paper by extending this result for $\alpha < 1/2$.

2. PRELIMINARY RESULTS

Let $\mathbb{G}(n,e)$ be the family of graphs with n vertices and e edges. We say that $G_* \in \mathbb{G}(n,e)$ is a *minimizer graph* if

$$f(G_*) = \min_{G \in \mathbb{G}(n,e)} f(G),$$

where $f(G) = \sum_{i=1}^{n} \sqrt{d_i}$, with $(d_1, \ldots, d_n)$ the degree sequence of G. We note that this definition does not imply the uniqueness of the minimizer graph.

As proven implicitly in [4], there is a strong connection between minimizer graphs (or, in a more general setting, between extremal graphs) and threshold graphs. Several equivalent properties may be used to characterize the threshold graphs; cf. [2,3,5]:

(i) there exists some hyperplane strictly separates the characteristic vectors of the independent sets of vertices of G from those of the non-independent sets.

(ii) every three distinct vertices V_i, V_j, V_k of G satisfy the following condition:

$$\text{if } d_i \le d_j \text{ and } V_iV_k \text{ is an edge, then } V_jV_k \text{ is an edge.}$$

(iii) G can be constructed from the one-vertex graph by repeatedly adding an isolated vertex or an universal one (a vertex adjacent to every other vertex);

The first property gave rise to the name "threshold graphs." They were first introduced in [2], and were studied quite extensively; see, for instance, the more than 150 references in [5]. This heightened interest is partly due to the fact that they arise naturally as potential solutions for a variety of extremal problems in graph theory.

In [4], it has been shown that every minimizer graph must be a threshold graph. For completeness, we include a proof of this result.

Lemma 2.1. *Every minimizer graph is a threshold graph.*

Proof. We use property (ii) to characterize the threshold graphs and give a proof by contradiction.

Let G_* be a minimizer graph in $\mathbb{G}(n,e)$ and assume that there exist three distinct vertices V_i, V_j, V_k of G_* such that $d_i \le d_j$, V_iV_k is an edge, and V_iV_k is not an edge of G_*. By replacing V_iV_k with V_jV_k we obtain a new graph $G' \in \mathbb{G}(n,e)$. We now prove that $f(G_*) - f(G') > 0$, which contradicts the fact that G was a minimizer graph.

All the vertices in G_* and G' have the same degrees, except for V_i and V_j. The degrees of V_i and V_j in G', respectively, are $d_i' = d_i - 1$ and $d_j' = d_j + 1$. Therefore,

$$\begin{aligned} f(G) - f(G') &= \sqrt{d_i} + \sqrt{d_j} - \sqrt{d_i'} - \sqrt{d_j'} \\ &= \sqrt{d_i} + \sqrt{d_j} - \sqrt{d_i - 1} - \sqrt{d_j + 1} \\ &= \frac{1}{\sqrt{d_i} + \sqrt{d_i - 1}} - \frac{1}{\sqrt{d_j + 1} + \sqrt{d_j}} \\ &> 0, \end{aligned}$$

since $d_i < d_j + 1$, by the assumption $d_i \le d_j$. ∎

In our paper, we only need the following result, which is a direct consequence of Lemma 2.1.

Corllary 2.2. *Assume that a minimizer graph $G \in \mathbb{G}(n,e)$ does not have isolated vertices, and let V_1 be a vertex of minimal degree of G. Then all the neighbors of V_1 have full degree.*

We approach the problem of finding the minimizer graphs for two different regimes, according to whether the graphs have "few" edges, i.e., $e \le \binom{n-1}{2}$ or have "many" edges, i.e., $e > \binom{n-1}{2}$.

In the first case, we show that any minimizer graph has at least one isolated vertex.

Lemma 2.3. *Let $G_* \in \mathbb{G}(n,e)$ be a minimizer graph, and assume that $e \le \binom{n-1}{2}$. Then G_* has at least one isolated vertex.*

Proof. Once again, we prove the result by contradiction. Assume that G_* does not have any isolated vertices, and let V_1 be a vertex of G_* of minimal degree $d_1 = k > 0$. Then V_1 has exactly k neighbors, denoted by $V_{n-k+1}, V_{n-k+2}, \ldots, V_n$. From Corollary 2.2, it follows that all these vertices are of full degree, that is, $d_{n-k+1} = \cdots = d_n = n-1$. Let $V_2, \ldots, V_{n-k}$, be the other vertices of G_*, of degrees $d_2, \ldots, d_{n-k}$, respectively. Since V_1 is of minimal degree, we obtain that

$$k \le d_i, \forall i = 2 \cdots n-k. \tag{1}$$

We construct a graph G' by deleting the k edges from V_1 and replacing them with k edges chosen arbitrarily in the subgraph spanned by $V_2, \ldots, V_{n-k}$. We note that this is possible, since the total number of edges is less than $\binom{n-1}{2}$. As before, we prove that $f(G_*) - f(G') > 0$, which contradicts the fact that G_* was a minimizer graph.

In G', the vertex degrees have changed as follows: $d_1' = 0$, $d_{n-k+1} = \cdots = d_n = n-2$, and $d_i' = d_i + \varepsilon_i$, where $\varepsilon_i \ge 0$ for all $i = 1, \ldots, n-k$ and

$$\sum_{i=2}^{n-k} \varepsilon_i = 2k, \tag{2}$$

since we have added k edges to the subgraph spanned by $V_2, \ldots, V_{n-k}$. Then,

$$\begin{aligned} f(G_*) - f(G') &= \sqrt{k} + \sum_{i=2}^{n-k} \sqrt{d_i} + k\sqrt{n-1} - \sum_{i=2}^{n-k} \sqrt{d_i + \varepsilon_i} - k\sqrt{n-2} \\ &\ge \sqrt{k} - \sum_{i=2}^{n-k} \left(\sqrt{d_i + \varepsilon_i} - \sqrt{d_i} \right) \\ &\ge \sqrt{k} - \sum_{i=2}^{n-k} \frac{\varepsilon_i}{\sqrt{d_i + \varepsilon_i} + \sqrt{d_i}}. \end{aligned} \tag{3}$$

From (1), it follows that

$$\frac{1}{2\sqrt{k}} \ge \frac{1}{\sqrt{k + \varepsilon_i} + \sqrt{k}} \ge \frac{1}{\sqrt{d_i + \varepsilon_i} + \sqrt{d_i}}. \tag{4}$$

Therefore, from (3) and using (2), we conclude that

$$f(G_*) - f(G') > \sqrt{k} - \sum_{i=2}^{n-k} \frac{\varepsilon_i}{2\sqrt{k}} = \sqrt{k} - \frac{1}{2\sqrt{k}} 2k = 0,$$

which contradicts the fact that G_* was chosen to be a minimizer graph. ■

It is interesting to note that there is a close relationship between minimizing the sum of the square roots of degrees over $\mathbb{G}(n, e)$ and maximizing the sum of the squares of degrees over $\mathbb{G}(n, e)$.

It has been shown in [1,6] that every optimal graph for the latter problem belongs to one of the six particular classes of threshold graphs. The complete characterization of the optimal graphs remains an open question.

However, for the case of very few edges, i.e., $e \leq n - 2$, we will show that, with a minor exception, the star with e edges (plus $n - e - 1$ isolated vertices) is the unique graph which maximizes the sum of the squares of the degrees.

We are going to need this result for a more general family of functions, so let us introduce the following notations. Given m a positive integer, $m \geq 2$, and a graph $G \in \mathbb{G}(n, e)$, let

$$F_m(G) = \sum_{i=1}^{n} (d_i)^m, \tag{5}$$

where $(d_1, \ldots, d_n)$ is the degree sequence of G. A graph $G^* \in \mathbb{G}(n, e)$ is a *maximizer graph* for F_m if

$$F_m(G^*) = \max_{G \in \mathbb{G}(n,e)} F_m(G).$$

Lemma 2.4. *Let $m \geq 2$ be a fixed integer, and let $\mathbb{G}(n, e)$ be the family of graphs with n vertices and e edges. If $e \leq n - 2$, then $G^* = K_{1,e} \cup S_{n-e-1}$, the star with e edges plus $n - e - 1$ isolated vertices, is a maximizer graph for F_m over $\mathbb{G}(n, e)$. Moreover, it is unique with this property, except the case when $m = 2$ and $e = 3$ (when both $K_{1,3} \cup S_{n-4}$ and $K_3 \cup S_{n-3}$ are maximizer graphs).*

Proof. Let G^* be a maximizer graph. It is easy to show that the vertex of maximal degree in G^* is connected with all the non–isolated vertices of G^*. If this were not the case, then we replace one of the edges corresponding to the non–isolated vertex by the edge joining it with the vertex of maximal degree. The argument is similar to that of Lemma 2.1.

Let V_n be the vertex of maximal degree, $d_n = p$, in the graph G^*. Then V_n is connected to p vertices, $V_1, \ldots, V_p$, and all the other vertices are isolated, i.e., $d_{p+1} = \cdots = d_{n-1} = 0$. Let

$$p \geq d_1 \geq d_2 \geq \cdots \geq d_p \geq 1 \tag{6}$$

be the degrees of $V_1, V_2, \ldots, V_p$. Since the total number of edges is e, the subgraph spanned by $V_1, \ldots, V_p$, has $e - p$ edges, and therefore

$$e - p \leq \binom{p}{2}.$$

We want to prove that $G^* = K_{1,e} \cup S_{n-e-1}$. It is enough to show that $d_1 = 1$. As before, we give a proof by contradiction, by finding a graph G' with $F_m(G^*) < F_m(G')$.

Assume that $d_1 > 1$. Starting with G^*, we construct a new graph G' by erasing the $d_1 - 1$ edges corresponding to V_1 from the subgraph with vertices $V_1, \ldots, V_p$ and replacing them with $d_1 - 1$ edges between V_n and $d_1 - 1$ isolated vertices. We note that we have sufficiently many isolated vertices to do this construction, since $e \leq n - 2$.

The vertex degrees of G' have changed as follows: $d'_1 = 1, d'_{p+1} = \cdots = d'_{p+d_1-1} = 1, d'_n = p + d_1 - 1$, and, for $2 \leq i \leq p$,

$$d'_i = \begin{cases} d_i - 1 & \text{for } d_1 - 1 \text{ vertices} \\ d_i & \text{for the other } p - d_1 + 1 \text{ vertices.} \end{cases} \tag{7}$$

We recall that $d_i \leq p$ for $1 \leq i \leq p$. Since $x^m - (x-1)^m$ is an increasing function for all positive integers $m \geq 2$, it follows that

$$d_i^m - (d_i - 1)^m \leq p^m - (p-1)^m,$$

and, from (7) that

$$\sum_{i=2}^{p} d_i^m - d_i'^m \leq (d_1 - 1)(p^m - (p-1)^m). \tag{8}$$

We use (8) to estimate $F_m(G^*) - F_m(G')$ as follows:

$$\begin{aligned} F_m(G^*) - F_m(G') &= d_1^m + \sum_{i=2}^{p} d_i^m + p^m - 1 - \sum_{i=2}^{p} d_i'^m - (d_1 - 1) - (p + d_1 - 1)^m \\ &\leq d_1^m + p^m - (p + d_1 - 1)^m - d_1 + (d_1 - 1)(p^m - (p-1)^m) \\ &= d_1(p^m - (p-1)^m - 1) - ((p + d_1 - 1)^m - (p-1)^m - d_1^m). \end{aligned}$$

To prove that $F_m(G^*) < F_m(G')$, it is then sufficient to show that

$$d_1(p^m - (p-1)^m - 1) < (p + d_1 - 1)^m - (p-1)^m - d_1^m.$$

For $m \geq 3$, the inequality above follows from binomial expansions:

$$d_1 \sum_{j=1}^{m-1} \binom{m}{j} (p-1)^j < \sum_{j=1}^{m-1} \binom{m}{j} (p-1)^j d_1^{m-j}.$$

For $m = 2$, using binomial expansions only shows that $F_m(G^*) \leq F_m(G')$. Therefore, we study the case when we have equality. By direct computation, we obtain

$$F_2(G^*) - F_2(G') = \sum_{i=2}^{p} (d_i^2 - d_i'^2) - (d_1 - 1)(2p - 1). \qquad (9)$$

If $d_i' = d_i - 1$, then $d_i^2 - d_i'^2 = 2d_i - 1 \leq 2p - 1$. From (7), it results that

$$\sum_{i=2}^{p} (d_i^2 - d_i'^2) \leq (d_1 - 1)(2p - 1). \qquad (10)$$

From (9) and (10), we obtain that $F_2(G^*) - F_2(G') \leq 0$ and $F_2(G^*) - F_2(G') = 0$ if and only if, for all indices $2 \leq i \leq p$ for which $d_i' = d_i - 1$, we have $d_i = p$.

We recall that $d_i' = d_i - 1$ is equivalent with saying that V_i was a neighbor of V_1. Hence, every neighbor of V_1 should have degree p, which gives that $d_1 = p$; cf. (6). This implies that all the vertices $V_2, \ldots, V_p$, are neighbors of V_1. By the above observation, it results that

$$d_1 = d_2 = \cdots = d_p = p.$$

Therefore, the only way our delete-add construction (performed under the assumption $d_1 > 1$) may fail to produce a graph G' which is *strictly better* than our initial graph G^* is when the only nontrivial component of G^* is a complete graph on $p + 1$ vertices.

In this case we can compare directly $G = K_{p+1} \cup S_{n-p}$ with our candidate $G^* = K_{1,e} \cup S_{n-e-1}$. We note that $e = \binom{p+1}{2}$, and obtain

$$F_2(G) = (p+1)p^2 \quad \text{and} \quad F_2(G^*) = \binom{p+1}{2}^2 + \binom{p+1}{2}.$$

This implies that:

$$F_2(G^*) - F_2(G) = \frac{(p-2)(p-1)p(p+1)}{4} \geq 0,$$

with equality if and only if $p = 1$ or $p = 2$. It is clear that, for $p = 1$, we obtain $K_{1,1} \approx K_2$. In conclusion, for $m = 2$ the only maximizer graph different from "star plus isolated vertices" is obtained for $p = 2$, i.e., when $e = 3$, and it is isomorphic to $K_3 \cup S_{n-3}$. ■

3. UNIQUENESS OF THE MINIMIZER GRAPH

We are now in position to prove the main result of our paper, the uniqueness of the minimizer graph for the sum of square roots of the vertex degrees.

Theorem 3.1. *Let $\mathbb{G}(n,e)$ be the family of graphs with n vertices and e edges. If $e = \binom{k}{2}$, then there is a unique minimizer graph $G_{1/2} \in \mathbb{G}(n,e)$ corresponding to $f(G) = \sum_{i=1}^{n} \sqrt{d_i}$ and it is isomorphic to the graph with a complete subgraph on k vertices and $n-k$ isolated vertices, i.e., $G_{1/2} = K_k \cup S_{n-k}$. Otherwise, let k be the unique positive integer such that $\binom{k-1}{2} < e < \binom{k}{2}$. Once again, there is a unique minimizer graph $G_{1/2} \in \mathbb{G}(n,e)$ which is isomorphic to the graph with $n-k$ isolated vertices, a complete subgraph K_{k-1}, and one vertex of degree $e - \binom{k-1}{2}$ connected to vertices of the complete subgraph.*

Proof. Let $G \in \mathbb{G}(n,e)$ be a minimizer graph. If $e = \binom{k}{2}$, we can use Lemma 2.3 to show that all the edges of G belong to a subgraph of G with k vertices. Since $e = \binom{k}{2}$, that subgraph is complete, and therefore there is exactly one minimizer graph, $G = K_k \cup S_{n-k}$.

Otherwise, let $k \le n$ be the unique positive integer such that

$$\binom{k-1}{2} < e < \binom{k}{2}. \tag{11}$$

Using Lemma 2.3 repeatedly, it follows that all the edges of G belong to a subgraph H with k vertices; the other $n-k$ vertices are isolated, i.e.,

$$G = H \cup S_{n-k}.$$

Since G is a minimizer graph over $\mathbb{G}(n,e)$, it follows that $H \in \mathbb{G}(k,e)$ must be a minimizer graph over $\mathbb{G}(k,e)$.

Let $(d_1,\ldots,d_k)$ be the degree sequence of H, and let $\bar{H}$ be the complement graph of H. Denote by $(\bar{d}_1,\ldots,\bar{d}_k)$, the degree sequence of $\bar{H}$, and by $\bar{e}$, the number of edges of $\bar{H}$. Since $\bar{d}_i = k-1-d_i$, it is easy to see, using (11), that

$$1 \le \bar{e} \le k-2. \tag{12}$$

We also note that H cannot have isolated vertices. Otherwise, the number of edges e, would be at most $\binom{k-1}{2}$, which contradicts (11). Therefore, $d_i \ge 1$, which implies that

$$\bar{d}_i = k-1-d_i \le k-2. \tag{13}$$

The main idea of the proof is to write $f(H)$ in terms of the degree sequence of the complementary graph $\bar{H}$, which can be done as follows:

$$f(H) = \sum_{i=1}^{k} \sqrt{d_i} = \sum_{i=1}^{k} \sqrt{k-1-\bar{d}_i} = \sqrt{k-1}\sum_{i=1}^{k} \sqrt{1 - \frac{\bar{d}_i}{k-1}}. \tag{14}$$

The Taylor expansion of the function $\sqrt{1-x}$,

$$\sqrt{1-x} = 1 - \frac{x}{2} - \sum_{m=2}^{\infty} \frac{(2m-3)!!}{2^m\, m!} x^m, \tag{15}$$

is absolutely convergent for every $0 < x < 1$, and is reduced to a trivial identity for $x = 0$. From (13), it follows that $0 \leq \frac{\bar{d}_i}{k-1} \leq \frac{k-2}{k-1} < 1$, and therefore we may use the expansion (15) for $\sqrt{1 - \frac{\bar{d}_i}{k-1}}$. To simplify notations, let $c_m = \frac{(2m-3)!!}{2^m\, m!}$. Then (14) can be written as

$$\begin{aligned} f(H) &= \sqrt{k-1} \sum_{i=1}^{k} \left(1 - \frac{\bar{d}_i}{2(k-1)} - \sum_{m=2}^{\infty} c_m \frac{\bar{d}_i^{\,m}}{(k-1)^m} \right) \\ &= \sqrt{k-1} \left[(k-1) - \frac{\bar{e}}{2(k-1)} - \sum_{m=2}^{\infty} \frac{c_m}{(k-1)^m} \sum_{i=1}^{k} \bar{d}_i^{\,m} \right]. \end{aligned}$$

Using the notation from (5), we obtain that

$$f(H) = \sqrt{k-1} \left(k-1) - \frac{\bar{e}}{k-1} - \sum_{m=2}^{\infty} \frac{c_m}{(k-1)^m} F_m(\bar{H}) \right). \tag{16}$$

We recall that $\bar{H} \in \mathbb{G}(k, \bar{e})$, and that $\bar{e} \leq k-2$; cf. (12). From Lemma 2.4, it results that the star with $\bar{e}$ edges, $K_{1,\bar{e}}$ is a maximizer of $F_m(H')$ over $H' \in \mathbb{G}(k, \bar{e})$, for every $m \geq 2$ (and is unique up to isomorphism for all $m \geq 3$). Since all the constants c_m are positive, it follows from (16) that $f(H)$ is minimal if and only if $\bar{H} = K_{1,\bar{e}}$.

Then, there is a unique minimizer graph H in $\mathbb{G}(k, \bar{e})$, and therefore a unique minimizer graph $G \in \mathbb{G}(n, e)$, with $G = H \cup S_{n-k}$. Since the complement of H is a star with $\bar{e} = \binom{k}{2} - e$ edges, the graph H has a complete subgraph K_{k-1}, and one vertex of degree $e - \binom{k-1}{2}$ connected to vertices of the complete subgraph. ■

4. EXTENSIONS OF THE MAIN RESULT

A natural generalization of our problem is to find the minimizer graphs over $\mathbb{G}(n, e)$ corresponding to the function

$$f_\alpha(G) = \sum_{i=1}^{n} (d_i)^\alpha,$$

with $0 < \alpha < 1$ a fixed exponent.

In the previous sections, we have studied the case $\alpha = 1/2$, and have found, for every given (n, e), a unique minimizer graph $G_{1/2}$ in $\mathbb{G}(n, e)$. In this section, we extend that result to the case $0 < \alpha < 1/2$, by following the same steps in the proof, and discuss a possible extension to the case $1/2 < \alpha < 1$.

The argument of Theorem 3.1 used for the case of graphs with many edges is valid for any $\alpha \in (0,1)$. The main reason is that the Taylor expansion of $(1-x)^\alpha$ has the form

$$(1-x)^\alpha = 1 - \alpha x - \sum_{m=2}^{\infty} c_{m,\alpha} x^m,$$

where all the coefficients $c_{m,\alpha}$ are positive for $0 < \alpha < 1$.

We note that Lemma 2.4 does not involve the exponent α, while Corollary 2.2 is valid for all $0 < \alpha < 1$, since the function $x^\alpha - (x-1)^\alpha$ is decreasing on $[1, \infty]$.

However, our proof for Lemma 2.3 is only valid for $0 < \alpha \leq 1/2$. To show this, we look at (3) and at (4). We obtain that

$$f(G) - f(G') \geq k^\alpha - \sum_{i=2}^{n-k} \left[(d_i + \varepsilon_i)^\alpha - d_i^\alpha \right].$$

Using the fact that $k \leq d_i$, for all $i = 2, \ldots, (n-k)$, and that the function $(x + \varepsilon_i)^\alpha - x^\alpha$ is decreasing on $[0, \infty]$ for $\varepsilon_i > 0$, it results that

$$(d_i + \varepsilon_i)^\alpha - d_i^\alpha \leq (k + \varepsilon_i)^\alpha - k^\alpha,$$

and therefore that

$$f(G) - f(G') \geq k^\alpha - \sum_{i=2}^{n-k} [(k + \varepsilon_i)^\alpha - k^\alpha]. \tag{17}$$

For any $\varepsilon_i > 0$, there exists a point δ_i in the open interval $(k, k + \varepsilon_i)$ such that

$$(k + \varepsilon_i)^\alpha - k^\alpha = \varepsilon_i \alpha \delta_i^{\alpha - 1}.$$

Then,

$$(k + \varepsilon_i)^\alpha - k^\alpha < \alpha k^{\alpha - 1} \varepsilon_i. \tag{18}$$

Using (2) and the inequality (18) to obtain a strict lower bound in (17), it follows that

$$f(G) - f(G') > k^\alpha - \sum_{i=2}^{n-k} \alpha k^{\alpha-1} \varepsilon_i = k^\alpha - \alpha k^{\alpha-1} 2k = (1 - 2\alpha) k^\alpha$$

In other words, if $\alpha \leq 1/2$, then $f(G) - f(G') > 0$, and the proof of Lemma 2.3 remains valid.

With all these ingredients in place, a proof similar to that of Theorem 3.1 can be given for the following more general result.

Theorem 4.1. *Let $\mathbb{G}(n, e)$ be the family of graphs with n vertices and e edges, and let $0 < \alpha \leq 1/2$. Let k be the unique positive integer such that $\binom{k-1}{2} < e \leq \binom{k}{2}$. Then there exists a unique minimizer graph $G_\alpha \in \mathbb{G}(n, e)$ and it is isomorphic to the graph with $n - k$ isolated vertices, a complete subgraph K_{k-1}, and one vertex of degree $e - \binom{k-1}{2}$ connected to vertices of the complete subgraph.*

An interesting open question is to decide what happens if $\alpha \in (1/2, 1)$. Numerical computations strongly suggest that the above result remains true.

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