Department of Mathematics *May 20, 2010*

Stochastic Processes for Finance (MTH 9862).

Final Examination.

Instructions: Please **print** your name below. Show all work and write legibly. Minimum to pass is 50 points. Full credit corresponds to 100 points. **Good luck!**

Student name:

Grade

Problem	Out of	Score	Comments
1	30		
2	20		6+6+8
3	30		
4	20		
5	25		
Total	125		

Problem 1. Let $B(t),\ t \geq 0$, be a standard Brownian motion (BM). Consider the following stochastic processes:

(1) The process $dD(t) = -rD(t)\,dt$, $D(0) = 1$, r is a positive constant.

(2) Geometric Brownian motion $S(t),\ t \geq 0$, such that $\log(S(1)/S(0))$ has a standard normal distribution, $S(0) = 20$.

(3) $dR(t) = \alpha(t)R(t)\,dt + \sigma(t)R(t)\,dB(t)$, $R(0) = 1$, where $\alpha(t)$ and $\sigma(t)$ are general bounded adapted processes.

(4) $W(t) = B_1(t)\sin\alpha + B_2(t)\cos\alpha$, where $(B_1(t), B_2(t))$ is a standard 2-dimensional Brownian motion and α is a real number.

(5) $Y(t) = \max\limits_{0\leq u\leq t} B(u)$.

(6) $Z(t) = \exp\left(\int_0^t \theta(u)\,dB(u) - \frac{1}{2}\int_0^t \theta^2(u)\,du\right)$, where $\theta(u),\ u \geq 0$, is a bounded measurable non-random function.

Fill out the following table. Possible answers are "yes" and "no". Answer "no" includes "sometimes". You do not need to justify your answers but if you use any computation, "educated guess", theorem, or definition to arrive at your answers, please, mention it. This might give you a partial credit where otherwise you would get 0. No points will be taken off for a failed attempt to explain a correct answer.

Property	$B(t)$	(1)	(2)	(3)	(4)	(5)	(6)
A.s. positive	no						
Standard BM	yes						
Gaussian	yes						
Ito process	yes						
Martingale	yes						
Markov	yes						
Regular	no						

Problem 2.

(a) State Girsanov's theorem (dimension 1).

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t \subset \mathcal{F}$, $t \geq 0$, be a filtration. Give a definition of a stopping time. State any one version of the optional stopping theorem.

(c) Give analytic and probabilistic characterizations of the price of a finite expiration American put option.

Problem 3. Let $B(t),\ t \geq 0$, be a standard Brownian motion and

$$S(t) = S(0)\, e^{\sigma B(t)+\mu t}, \quad t \geq 0,$$

where $S(0)$ is a given constant, μ and $\sigma \neq 0$ are real numbers.

(a) Find $dS(t)$ and use it to compute $E\big(S(t)\big)$.

(b) Find $d\big((S(t))^2\big)$ and use it to compute $\mathrm{Var}(S(t))$.

(c) Find $d(S(t)B(t))$ and use it to compute $\mathrm{Cov}(S(t), B(t))$.

(d) Find the *correlation*, $\rho(S(t), B(t))$, between $S(t)$ and $B(t)$.

Problem 4. Let $B(t),\ t \geq 0$, be a standard Brownian motion and $X(t)$, $t \geq 0$, be the solution of the equation ($\sigma > 0$ is a constant)

$$dX(t) = (1 - X(t))\, dt + \sigma\, dB(t), \quad X(0) = 1.$$

Suppose that $T > 0,\ r \geq 0$ are fixed and for $t \in [0, T]$

$$V(t) = \mathbb{E}(e^{-r(T-t)} h(X(T), B(T)) | \mathcal{F}(t)),$$

where $h(x, y)$ is a bounded Borel function on $\mathbb{R}^2$ and $\mathcal{F}(t),\ t \geq 0$, is the filtration generated by the Brownian motion. Explain why $V(t)$ can be written as $v(t, X(t), B(t))$ and find the partial differential equation and the terminal condition for $v(t, x, y)$.

Problem 5. Assume Black-Scholes-Merton model. An American cash-or-nothing option can be exercised at any time $t \in [0, T]$. If exercised at time t its payoff is

$$\begin{cases} 1, & \text{if } S(t) \leq K; \\ 0, & \text{if } S(t) > K. \end{cases}$$

(a) What is the optimal exercise strategy? Describe the stopping set and the continuation set for this problem.

(b) Let $r = 0$. Let $v(0, x)$ be the time 0 value of this option when $S(0) = x$. Find $v(0, x)$ if you are given $f_\alpha(t, x)$, the density of the process

$$Y(t) := \max_{0\leq s\leq t} (\alpha s + \tilde{B}(s)),$$

where $(\tilde{B}(s))_{s\geq 0}$ is the standard Brownian motion under the risk-neutral measure and α is a real number.