

Stochastic Processes for Finance (MTH 9862).

Midterm Examination.

Instructions: Please **print** your name below and submit this page as a cover page. Please show all work and write legibly. Please give references to *every* source (books, discussions, hints, etc.) that you are using. Use of Internet sources is not allowed. **Good luck!**

Student name:

Grade

Problem	Out of	Score	Comments
1	20		
2	15		
3	15		
4	15		
5	20		
6	15		
Total	100		

Problem 1. Let $X_i, i = 1, 2, \dots$, be i.i.d. random variables, and

$$P(X_i = 1) = 1 - P(X_i = -1) = p, \quad p \in (0, 1), \quad i = 1, 2, \dots$$

Fix $x \in \mathbb{Z}$ and consider a simple random walk

$$S(0) = x, \quad S(n) = x + \sum_{i=1}^n X_i, \quad n = 1, 2, \dots$$

The walk starts at x and moves up or down by one unit. For $m \in \mathbb{Z}$ define $T_m = \inf\{n > 0 : S(n) = m\}$. Let $a < x < b$, where a and b are integers and $T = T_a \wedge T_b$ (the first exit time from the strip (a, b)).

- (a) Let $c = b - a$. Argue that $P(T > c) < 1 - p^c$.
- (b) Use conditioning to show that

$$P(T > nc) < (1 - p^c)P(T > (n - 1)c), \quad n = 1, 2, \dots,$$

and conclude that $P(T > nc) < (1 - p^c)^n$.

- (c) Show that $P(T < \infty) = 1$.

From now on let $p = 1/2$.

- (d) Show that $S(n \wedge T)$ and $S^2(n \wedge T) - (n \wedge T)$, $n = 1, 2, \dots$, are martingales.
- (e) Use $S(n \wedge T)$ and the Dominated Convergence Theorem to compute $P(T_a < T_b)$ and $P(T_b < T_a)$.
- (f) Use (e) to show that for every starting point x and every $m \in \mathbb{Z}$

$$P(T_m < \infty) = 1.$$

- (g) Use $S^2(n \wedge T) - (n \wedge T)$ and the Monotone Convergence Theorem to find ET .

Problem 2. Let $W(t), t \geq 0$, be the standard Brownian motion under \mathbb{P} . Define $M(t) = tW(1/t)$ for $t > 0$ and $M(0) = 0$. Show that $M(t), t \geq 0$, is a Brownian motion under \mathbb{P} . Note: do not forget to show the continuity at $t = 0$ (the rest is very easy).

Problem 3. Let $W(t)$, $t \geq 0$, be a standard Brownian motion and $\sigma > 0$ be a constant. Suppose that a stock price $S(t)$, $t \geq 0$, satisfies the equation

$$dS(t) = \sigma S(t) dW(t) + \frac{1}{2} \sigma^2 S(t) dt.$$

Consider a barrier call option on this stock with the strike price K and expiration $T > 0$, which becomes alive only if $\min_{0 \leq t \leq T} S(t) \leq m$, where $m < \min\{K, S(0)\}$ is a given barrier. In other words, the payoff of this option is equal to $(S(T) - K)_+$ if $\min_{0 \leq t \leq T} S(t) \leq m$ and 0 if $\min_{0 \leq t \leq T} S(t) > m$. Find the probability that this option will finish in the money.

Problem 4. Suppose that

$$dS(t) = \sigma S(t) dW(t), \quad S(0) = s,$$

where $W(t)$, $t \geq 0$ is Brownian motion and s is a positive constant. Find the mean and variance of the process

$$Y(t) = \frac{1}{t} \int_0^t S(u) du.$$

Problem 5. The Black-Karasinski model of interest rates is

$$dR(t) = \sigma(t)R(t) dW(t) + \left(\theta(t) + \frac{1}{2} \sigma^2(t) - \alpha(t) \log R(t) \right) R(t) dt,$$

where $W(t)$, $t \geq 0$ is a standard Brownian motion and $\sigma(t)$, $\theta(t)$, and $\alpha(t)$ are deterministic functions of time. In the special case where σ , θ , and α are constants, find $R(t)$ as a function of $\int_0^t e^{\alpha s} dW(s)$.

Problem 6. Solve problem 5.11 from Shreve, vol. II.