MTH 9814
Final Exam
Spruill

Directions:
This packet contains 10 questions. Your assignment is to work 8 of them.

4 questions will be worth a maximum of 15 points apiece; 4 questions will be worth a maximum of 10 points apiece. The choice of which 4 questions receive the higher weight is yours. You must indicate on this cover sheet which 4 problems should be allocated 15 points each, and which 4 should be allocated 10 points each.

In addition, you may work a ninth problem of your choice, worth a maximum of 10 points of extra credit.

Only responses to the questions **indicated in your point allocations** will be graded. All other work will be ignored. You are encouraged to do scratch work separately and record polished answers on the question pages. Additional sheets may be attached if needed, but please **clearly indicate** the problem with which any additional work is associated.

Pencils, pens, calculators, and a crib sheet (1 page front and back) are permitted for this exam. No other resources may be used.

Record answers to 5 decimal places.

Show enough work that I can follow your reasoning. Sound reasoning with errors in the final calculation will receive partial credit. A numerical answer with no other supporting explanation will **not** receive full credit.

Point allocations:
Allocate 15 points each to:
      Problem #_____
      Problem #_____
      Problem #_____
      Problem #_____

Allocate 10 points each to:
      Problem #_____
      Problem #_____
      Problem #_____
      Problem #_____

Extra Credit (10 points):
      Problem #_____

Problem #1

The term structure of interest rates is as follows:

t (years)	r (continuous)
0.25	0.015
0.5	0.02
1	0.025
2	0.023

(a) What is the discount factor at the maturity $t = 0.5$ years?

(b) What is the zero rate at maturity $t = 0.25$ years expressed with semiannual compounding?

(c) What is the forward rate from $t = 1$ to $T = 2$ years expressed as a rate of simple interest?

(d) Using linear interpolation in the continuously compounded zero rate, determine the par yield of a bond paying semiannual coupons, expressed with semiannual compounding.

Problem #2

A portfolio consists of the following:

  A long zero-coupon bond position paying 5
  A long put position struck at 20
  Two long call positions struck at 20
  Two short call positions struck at 30
  A long put position struck at 35

All options are European and written on the same underlying; the maturity of the bond and the expiry time of all options are the same.

On a well labeled graph, plot the payoff of this portfolio at option expiry versus the terminal spot price of the underlying in the range 0-40.

Problem #3

A bond paying a coupon rate of 5.5% with semiannual coupons has a maturity of 2 years. The yield of the bond is 4.5% expressed with continuous compounding.

(a) What is the price of the bond, expressed per 100 notional?
(b) What is the (Macaulay) duration of the bond?
(c) What is the convexity of the bond?
(d) What is the modified duration of the bond?

Problem #4
For the following questions, the risk-free interest rate is constant at 4%, expressed with continuous compounding.

(a) What is the one-month forward price of an asset trading at 40 today if it pays no dividends?
(b) What is the two-month forward price of an asset trading at 100 today if it pays a continuous dividend at a rate of 2.5%?
(c) What is the three-month forward price of an asset trading at 25 today if it pays a fixed dividend of 2 in one month and 2.5 in two months?
(d) Options with a one-year expiry trade on a particular exotic asset. The price of a European call option on this asset struck at 35 is 3.24452; the price of a European put option on this asset struck at 35 is 4.68570. What is the option-implied one-year forward price of the asset?

Problem #5

The term structure of discount factors is as follows:

t (years)	discount factor
0.5	0.975309912
1	0.955997482
1.5	0.927743486
2	0.900324523

(a) What is the par swap rate to 2 years when payments on the fixed leg occur annually?
(b) What is the par swap rate to 2 years when payments on the fixed leg occur semiannually?
(c) What is the forward par swap rate for a swap that commences in 1 year and has a tenor of 1 year if payments on the fixed leg occur semiannually?
(d) What is the present value of a 1-year swap with notional 100 commencing in 1 year if the holder pays a fixed rate of 2.5% and payments occur semiannually?

Problem #6

An option portfolio consists of the following positions:

position	quantity	option delta	option gamma	option vega
Long Call	100	0.461	0.02	0.198
Long Put	35	-0.249	0.016	0.158
Short Call	70	0.291	0.017	0.171

The following hedging instruments are available in the market:

position	quantity	option delta	option gamma	option vega
hedge 1	1	0.371	0.019	0.189
hedge 2	1	-0.439	0.02	0.197

Additionally, the underlying can be traded for hedging purposes.

For the following questions, assume that the hedging instruments and the underlying can be traded in any quantity, including fractional quantities.

(a) Determine the aggregate delta, gamma, and vega of the portfolio.
(b) Determine the quantities of hedge 1 and underlying that, if added to the portfolio, make it gamma- and delta-neutral.
(c) Determine the quantities of hedge 2 and underlying that, if added to the portfolio, make it vega- and delta-neutral.
(d) Determine the quantities of hedge 1, hedge 2, and underlying that, if added to the portfolio, make it gamma-, vega-, and delta-neutral.

Problem #7

A risky issuer has a continuous-time hazard rate $\lambda = 0.01$ to all maturities. The risk-free interest rate is 5% to all maturities, expressed with continuous compounding.

(a) What is the unconditional probability of default of the issuer between now and 6 months from now? Between 6 months from now and a year from now?

(b) What is the present value of a bond from this issuer with a 1-year maturity that pays a coupon of 5.8% semiannually? Assume 100 notional and no recovery in the event of default.

(c) Make the simplifying assumption that default, if it occurs, can only occur at $t = 0.5$ years or $t = 1$ year, and that the default probabilities for each period are as above in (a). If the recovery rate in the event of default is 40%, what is the present value of the bond in (b)? (Assume that, if default occurs at a time when a bond payment is due, then the payment is *not* made. The recovery is only on the principal amount, not on any accrued coupon of the bond.)

(d) Under the default model above in (c), what is the one-year par CDS spread for this issuer? (Assume that, in the event of default, the protection buyer must pay the accrued premium up to the time of default.)

Problem #8

The binomial tree method for option pricing was initially proposed by Cox, Ross, and Rubinstein, who determined the up and down step sizes of the tree by setting:

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = \frac{1}{u}$$

…where σ is the annual volatility of the underlying and $\Delta t = T / N$, with T the time to expiry of the option in years, and N the number of time steps in the binomial model.

For this problem, we consider an underlying that pays no dividends under the following market conditions:

$S_0 = 100$

$\sigma = 0.5\%$

$r = 6.5\%$ (expressed with continuous compounding)

We wish to use the CRR parameterization to price an option on this underlying with an expiry of 0.75 years.

(a) Calculate the size of up and down steps u and d and the values of the transition probabilities p_u and p_d if we price the option on a tree with $N = 250$ time steps.

(b) The CRR parameterization is not unconditionally stable: In particular, this model is apt to fail for underlying assets with low volatility when the number of time steps is small. To illustrate this fact, show that the CRR parameterization is not arbitrage-free in the case above if $N = 100$ time steps.

(c) Assuming an asset that pays no dividends, develop a general expression for the minimum number of time steps N_{min} necessary to price an option with expiry T on an underlying with volatility σ when the continuously compounded risk-free rate is r.

(d) Use your expression above in (c) to calculate N_{min} for this particular case, and verify that the resulting model is arbitrage-free.

Problem #9

The current exchange rate between currency A and currency B is 1.25 units of currency B per unit of currency A. The risk-free rate to all maturities is 3% in currency A and 5% in currency B, both expressed with continuous compounding.

You are an investor whose domestic currency is currency A; you are long a zero-coupon bond that pays 100 units of currency B in 6 months.

(a) What is the present value of the bond, expressed in your domestic currency A?

(b) You enter into an FX forward agreement to pay 100 in currency B and receive a fixed amount in currency A in 6 months. What is the fair forward exchange rate, expressed in units of currency B per unit of currency A?

(c) After you enter into the agreement, currency B strengthens dramatically so that the current exchange rate is now 1.2 units of currency B per unit of currency A. What is the change in present value of the bond, expressed in your domestic currency A?

(d) Under the scenario above in (c), what is the change in present value of the entire portfolio (bond + FX forward)?

Problem #10
A European binary call option pays 1 if the price S_T of the underlying at expiry is greater than the strike K, and 0 otherwise. We designate the present value of a European binary call $C_B(S_0, K, r, q, \sigma, T)$ and the price of a vanilla European call $C(S_0, K, r, q, \sigma, T)$. (For the sake of notational cleanliness, we will henceforth omit the variables on which these valuation formulas depend.)

(a) It is not difficult to show that the risk-neutral valuation formulas of the two instruments are related by:

$$C_B = -\frac{\partial C}{\partial K}$$

Use this fact and the Black-Scholes pricing formula for a vanilla call option to prove that, under the Black-Scholes assumptions, the risk-neutral value of a European binary call is given by:

$$C_B = e^{-rT} N(d_2)$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

(Hint: You will need to use the fact, first seen in the derivation of delta of a call option, that:

$$Ke^{-rT} e^{-\frac{1}{2}d_2^2} = S_0 e^{-qT} e^{-\frac{1}{2}d_1^2})$$

(b) Assume that $r = q = 0$, and use the above to find the resulting valuation formula for an at-the-money European binary call (i.e., one for which $K = S_0$). What is the value of this option as the volatility approaches zero? As it approaches infinity?

(c) What is the sign of vega for the option above in (b)?