MTH 9831 Assignment 2 (09/11 - 09/17).

Read Lecture notes 2. Some additional references for this material are:

1. A. Etheridge. A Course in Financial Calculus. Sections 1.5, 1.6. (One period models.)
2. S. Ross. An Elementary Introduction to Mathematical Finance. pp. 104-106. (Delta hedging.)
3. A.Černý. Mathematical Techniques in Finance. Tools for Incomplete Markets. Chapter 2.(One period model.)
4. S. Shreve, Stochastic Calculus for Finance I, p. 140 (the last problem).

Solve:

(1) Consider a 3-period binomial model with parameters u, d, r such that $0 < d < 1+r < u$. Assume that the initial stock price is s. We would like to hedge and price a contingent claim which expires at time 3 and pays $x_{i,3}$ if $S(3) = u^i d^{3-i} s$, $i = 0, 1, 2, 3$. Starting from time 3 and going "back in time" construct replicating portfolios at each node of the tree at times $t = 2, 1, 0$. Conclude that every contingent claim, whose payoff depends only on $S(3)$, can be hedged and that the value of the portfolio that you obtained for time 0 is the unique no-arbitrage price of this claim. What is the no-arbitrage price of this claim if one wants to sell it at time 2? at time 1?

Find explicitly replicating portfolios and the price of the claim at each node for $t = 2, 1, 0$ if $u = 1.25$, $d = 0.8$, $r = 0.1$, $s = 36$, and $x_{i,3} = 2|i - 1.5|$, $i = 0, 1, 2, 3$.

The above construction is called "delta-hedging". The "delta" of the portfolio at each note is the number of shares of the underlying asset in the replicating portfolio at that node.

(2) Consider the following 1 period market with 3 assets:

- Asset 1: Money Market Account paying 5% per period.
- Asset 2: stock with initial price $S_0^1 = 1$ per share with possible final prices 1.2 and 0.8.
- Asset 3: another stock with initial price $S_0^2 = 1$ per share with possible final prices 1.2 and 0.8.

(a) Find the number n of possible states of this market.

(b) Describe all state price vectors and all risk-neutral probabilities.

(c) Consider the contingent claim C with payoff $C = (C_1, \dots, C_n)$, where n is the number of possible states. What are the smallest V_- and the largest V_+ prices permitted by the arbitrage considerations?

(d) Which claims C are attainable? What is the price of an attainable claim?

(e) Consider a digital call option, which pays 1 at time 1 if the average price of the stocks at time 1 is above 1.05 and 0 otherwise. Find the no-arbitrage price (or range of prices) for this call.

(3) Consider a trinomial one period model: if the time 0 price of the stock is s then at time 1 there are 3 possible prices, $1.2s$, s, and $0.8s$. The market consists of a riskless asset with one period interest rate 5% and the stock. This market is incomplete.

(i) Find the range of no-arbitrage prices for a call option on the stock with strike $0.9s$.

(ii) Add a call option with strike $0.9s$ and expiration 1 to the other two assets. Suppose that its time 0 price is $C = cs$. When is the market consisting of these three assets arbitrage-free? When is it complete? Give precise conditions.

(iii) Suppose that $C = 0.15s$. What can you say about the no arbitrage price of the call option with strike $1.05s$?

(4) (Joint distribution of random walk and its maximum-to-date.) Let W_n be a simple symmetric random walk. Define

$$W_n^* = \max_{1 \le j \le n} W_j.$$

For $m \le n$ and $2m - b \le n$ compute $P(W_n^* \ge m,\ W_n = b)$ using an argument based on reflected paths.