MTH 9831 Assignment 11 (11/25 - 12/03).

Read Lecture notes 12. Additional references for this material are:

- J. Jacod, Ph. Protter, Probability Essentials, Chapters 24, 25.
- A. Etheridge, A Course in Financial Calculus, Sections 2.3, 2.4.
- A. N. Shiryaev, Probability, Chapter 7, Sections 1 and 2.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, $\mathcal{F}_n \subset \mathcal{F}$, $n \geq 0$. All martingales and stopping times below are with respect to this filtration unless stated otherwise.

Solve:

(1) Let T and S be stopping times. Show that $T \wedge S$, $T \vee S$, and $T + S$ are stopping times.

(2) Let T be a stopping time. Define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}.$$

Show that $\mathcal{F}_T$ is a σ-algebra and that T is $\mathcal{F}_T$-measurable. The meaning of $\mathcal{F}_T$ is that it is information available up to a (random!) time T.

(3) Show that if S and T are stopping times such that $S \leq T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.

(4) Consider a binomial model with parameters u, d, r; $d < 1 + r < u$. Let p be the probability of the stock to go up in any one period (stock movements are assumed to be independent) and S_n be the stock price at time n.

(a) Show that the discounted stock price process $(\tilde{S}_n)_{n\geq 0}$, where $\tilde{S}_n = (1+r)^{-n} S_n$, is a martingale (with respect to its natural filtration) if and only if $p = (1 + r - d)/(u - d)$.

(b) Let $p = (1 + r - d)/(u - d)$. We know that the discrete stochastic integral of a bounded predictable sequence with respect to $(\tilde{S}_n)_{n\geq 0}$ is again a martingale (pp. 6-7 of LN12). The converse of this statement also holds and is known as the binomial representation theorem: Let $(\tilde{V}_n)_{n\geq 0}$ be a martingale with respect to the natural filtration of $(\tilde{S}_n)_{n\geq 0}$. Then there is a predictable process $(H_n)_{n\geq 1}$ such that

$$\tilde{V}_n = \tilde{V}_0 + \sum_{i=1}^{n} H_i(\tilde{S}_i - \tilde{S}_{i-1}).$$

Check that the process defined by

$$H_n(\omega_1\omega_2 \ldots \omega_{n-1}\omega_n) = \frac{V_n(\omega_1\omega_2 \ldots \omega_{n-1}u) - V_n(\omega_1\omega_2 \ldots \omega_{n-1}d)}{S_n(\omega_1\omega_2 \ldots \omega_{n-1}u) - S_n(\omega_1\omega_2 \ldots \omega_{n-1}d)}$$

satisfies the requirements. Here $(\omega_1\omega_2\dots\omega_n)$ represents a path on the n-step binomial tree, $\omega_i \in \{u, d\}$.

The significance of this representation is that if we think of $\tilde{V}_n$ as the discounted value of a contingent claim at time n then the process $(H_n)_{n\geq 1}$ is the hedging strategy (H_n is the number of shares of the underlying stock in the replicating portfolio to be held over the n-th period). So this proves that every contingent claim can be hedged and tells you exactly how (in this model).

(5) Let $(W_n)_{n\geq 0}$ be a simple random walk (symmetric or asymmetric) that starts from 0.

(a) Fix $\theta > 0$ and let $m(\theta) = \ln \mathbb{E}(e^{\theta W_1})$. Show that for every fixed $\theta > 0$

$$M_n = e^{\theta W_n - nm(\theta)}$$

is a martingale.

(b) Use the optional stopping theorem (give a careful argument) to compute the Laplace transform of $\tau_b = \inf\{n > 0 : W_n = b\}$, $b \in \mathbb{N}$, that is $\mathbb{E}\left(e^{-\lambda\tau_b}\right)$, $\lambda \geq 0$.

(c) Use part (b) to find $\mathbb{P}(\tau_b < \infty)$ and $\mathbb{E}\tau_b$. You will need to consider separately 3 cases: $\mathbb{E}W_1$ is positive, negative, or zero.